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Declaration and Acknowledgement

This thesis contains an original presentation of the work performed either jointly with, or under the direction and guidance of my supervisor C.T.H. Baker. Some of the material has been presented in joint papers with C.T.H. Baker:

- (i) Stability analysis of Runge-Kutta methods applied to a linear Volterra integral equation. J.Austral.Math.Soc. (B) 22, pp.515-538 (1981).
- (ii) On the construction of stability polynomials for modified R-K methods for Volterra integro-differential equations. Contributions to "Treatment of Integral Equations by Numerical Methods" (eds. BAKER, C.T.H. & MILLER, G.F.), Academic Press (1982).

No part of the work has been submitted for any other degree or qualification.

ABSTRACT

Stability in the numerical treatment of Volterra integral and integro-differential equations with emphasis on finite recurrence relations.

In the last two decades the theory of Volterra integral equations and of integro-differential equations has developed extensively. New classes of methods for the numerical solution of such equations have been developed and at the same time there have been advances in the qualitative theory of these equations. More frequent use is being made of Volterra equations to model various physical and biological phenomenon as new insight has occurred into the asymptotic behaviour of solutions. In consequence, there has emerged a need for reliable and efficient methods for the numerical treatment of such equations.

This thesis is concerned with an aspect of numerical solution of Volterra integral and integro-differential equations. In Chapters 1 and 2 we are concerned with background material. We provide results on the classical theory of Volterra equations in Chapter 1 and on numerical methods in Chapter 2. The original material is contained in Chapters 3, 4 and 5. Here, stability results which involve the construction and analysis of finite-term recurrence relations are presented. The techniques relate to the treatment of Volterra integral and integro-differential equations. They permit the analysis of classical and γ -modified numerical methods.

The results presented should be viewed as a contribution towards an understanding of numerical stability for the methods

considered. The area is one in which further work (subsequent to the present investigation and involving advanced techniques) has been performed and where open questions still remain.

The techniques which are employed in this thesis are applicable in other areas of numerical analysis and therefore have intrinsic interest.

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CHAPTER 1

1.1 Preliminary Remarks

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1.2.2 Equations of the second kind

1.2.3 Equations of convolution type

1.3 Transformation of Equations into New Forms

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1.1 Preliminary Remarks.

In this Chapter we give some background material. Asterisked sections are peripheral to our main theme.

One of the first evolutionary integral equations mentioned in mathematical literature appears to be due to N.H. Abel who generalised Christian Huygens' investigations of the isochronous pendulum. Abel started from a problem in mechanics which was to determine, in a vertical plane, the path along which a particle must be constrained to fall under constant vertical acceleration so that its time of fall shall always equal a prescribed function of the distance fallen. If the particle falls, without friction, from a given height h to $y < h$, then from energy considerations we have $\frac{1}{2}m\left(\frac{ds}{dt}\right)^2 = mg(h - y)$ and

$$\int_{y=h}^{y=0} \frac{ds}{\sqrt{h - y}} = \sqrt{2g}T(h)$$

where $T(h)$ is the time taken to fall through the distance h .

Introducing the function $u(y)$ where $-ds/dy = u(y)$ we obtain Abel's equation

$$\int_0^h \frac{u(y)dy}{\sqrt{h - y}} = \sqrt{2g}T(h).$$

Huygens, in 1673, discovered the isochrone and tautochrone properties of the cycloid which were the solutions to his problem. Abel, by rather different investigations, published his solution (see §1.2 below) in 1826 in Crelle's Journal.

In 1896, V. Volterra (1860-1940), one of the founders of the modern theory of integral equations, published his theory of Integral Equations using their solution as a problem of finding the inverses of certain integral operators; while in 1900 Ivor Fredholm made a contribution on integral equations which had great impact on the foundation and development of functional analysis.

Abel equations occur frequently in the literature and are of great interest. For example, many are derived from problems in heat transfer and are related to solutions of certain parabolic equations. However, it is in the last two decades that the theory of Volterra integral equations has developed extensively. This may be because such equations are now seen to result from specific problems and are no longer regarded as special cases of Fredholm equations. As an illustration of the wealth of recent literature assessed in the study of Volterra integral equations we may refer to Tsalyuk [51], who, in a survey covering 1966-1976, reviewed 515 papers, none having an applied character, and, furthermore, he restricted his investigation to include only those which were reviewed in the Referativnyi Zhurnal "Matematika". Equally, the applications of Volterra integral equations are numerous and varied in many areas of science. For example, the analysis of problems of industrial replacement is similar to that in population analysis with its related actuarial and genetic applications cf. Lotka [36], [37] and Feller [25]. Furthermore, a recent monograph Brunner & van der Houwen [12], provides a "state of the art" in the numerical solution of such equations.

Concerning integral equations in general, Lonseth [35] has provided a useful survey of other applications of integral equations including problems in geophysics, hereditary phenomena in physics and biology, quantum mechanics, radiation, automatic control systems and communication theory. For further discussion of other sources and applications particularly of Volterra equations we may refer to Miller [41], Noble [42], Saaty [47], Tricomi [50], Hethcote et al [28], Feller [25], Lotka [36], Diekmann [22], Levin & Nohel [33], Davis [21], Bellman & Cooke [7], Zemanian [57] and, most recently, Burton [13].

In the discussion which follows we shall describe an integral equation as a functional equation which involves integrals of an unknown

function $f(x)$. For our purposes we assume that $f(x)$ is a real- or complex-valued function of a single variable x ; systems of equations can be considered if we extend the notation to vector-valued functions.

1.2 Classification of Abel-type and Volterra Integral Equations

The "kernel" $H(x, y, v)$, to which we refer below, denotes a prescribed function which is assumed for simplicity to be continuous for $-\delta \leq y \leq x + \delta \leq X + \delta$ and $|v| < \infty$ for some $\delta > 0$, $X < \infty$.

Furthermore, $g(x)$ denotes a prescribed function on $[-\delta, X + \delta]$.

The classical Abel equation, referred to in the introductory paragraph, is the equation

$$\int_0^x \frac{f(y)}{(x-y)^\alpha} dy = g(x), \quad (0 < \alpha < 1), \quad (1.1)$$

where α is given, and which may be called ill-posed. Under appropriate restrictions on $g(x)$, it has the solution

$$f(x) = \frac{\sin \alpha \pi}{\pi} \frac{d}{dx} \int_0^x \frac{g(y)}{(x-y)^{1-\alpha}} dy$$

(cf. (1.68)). The case $\alpha = 0$ is mentioned below.

1.2.1 Equations of the first kind.

Equation (1.1) may be generalised to produce a wide class of 'equations of Abel type'. The generalised Abel equation of the first kind, in which the solution $f(x)$ is sought, is of the form

$$\int_0^x \frac{H(x, y, f(y))}{(x^p - y^p)^\alpha} dy = g(x), \quad (0 < \alpha < 1, p = 1 \text{ or } 2, \text{ given}). \quad (1.2)$$

The linear case of equation (1.2),

$$\int_0^x \frac{K(x, y)f(y)}{(x^p - y^p)^\alpha} dy = g(x), \quad (0 < \alpha < 1, p = 1 \text{ or } 2, \text{ given}), \quad (1.3)$$

where $H(x, y, v) = K(x, y)v$ and $K(x, y)$ is continuous at least for $0 \leq y \leq x$, is also of interest. We refer to Anderssen and de Hoog [2] for further comments.

If, in equation (1.2), $\alpha = 0$ and $H(x, y, f)$ is a smooth kernel

we obtain a genuine Volterra problem resulting in a Volterra equation of the first kind

$$\int_0^x H(x, y, f(y)) dy = g(x), \quad x \in [0, X], \quad (1.4)$$

where X may be taken arbitrarily large. A basic example of (1.4) is the equation

$$\int_0^x f(y) dy = g(x).$$

On differentiating, we obtain the solution $f(x) = g'(x)$ provided $g'(x)$ exists and $g(0) = 0$. From (1.4) with $x = 0$ we see $g(0) = 0$.

Thus, we see that the equation only has a solution if $g(x)$ has special properties. This suggests that care must be taken lest the problem in solving (1.4) is 'improperly posed'.

1.2.2 Equations of the second kind.

The corresponding Abel and Volterra equations of the second kind, are respectively,

$$f(x) - \int_0^x \frac{H(x, y, f(y))}{(x^p - y^p)^\alpha} dy = g(x), \quad (0 < \alpha < 1, p = 1 \text{ or } 2, \text{ given}) \quad (1.5)$$

and

$$f(x) - \int_0^x H(x, y, f(y)) dy = g(x), \quad x \in [0, X] \quad (1.6)$$

In many Volterra equations (1.6), when $\alpha = 0$, it is found that, provided the kernel is sufficiently smooth, the smoothness of the solution f depends on the smoothness of g . It will be observed that the Abel equations (1.2), (1.3) and (1.5) exhibit weak singularities in the integrand. It is known that solutions of Abel equations frequently demonstrate bad behaviour near $x = 0$ even when $g(x)$ is smooth and later we shall show that the solution $f(x)$ of (1.5) may be expected to have a weak singularity despite $g(x)$ being well behaved.

Remark

Equations which arise from practical problems may have forms which differ slightly from those above. For example, Volterra equations of the second kind may be of the form

$$f(x) - \int_a^x H(x, y, f(y))dy = g(x), \quad x \in [a, X] \quad (1.7)$$

which is not apparently of the form of (1.6). However, we may use (1.6), without loss of generality, as we now illustrate. Using a linear change of variables in (1.7) and taking $\varphi(x) = f(x + a)$ we obtain

$$\varphi(x) - \int_a^{a+x} H(x + a, y, f(y))dy = g(x + a)$$

$$\text{that is, } \varphi(x) - \int_0^x H(x + a, y + a, \varphi(y))dy = g(x + a), \quad 0 \leq x \leq X' = X - a. \quad (1.8)$$

Equation (1.8), for the function $\varphi(x)$, has the form of equation (1.7) with $a = 0$. Conversely, given $\varphi(x)$ as the solution of (1.8) we may show that $f(x) = \varphi(x - a)$ is the solution of (1.7). Thus, the constant a in (1.7) is usually taken to be zero and in the following work we shall take (1.6) to be our general non-linear Volterra integral equation of the second kind.

1.2.3 Equations of Convolution Type.

Special subclasses of integral equations occur often. The kernel $H(x, y, f)$ is known as a convolution kernel if $H(x, y, f) = h(x - y; f)$ and non-linear Volterra integral equations of convolution type of the first and second kinds, are, respectively,

$$\int_0^x h(x - y; f(y))dy = g(x) \quad (1.9)$$

and

$$f(x) - \int_0^x h(x - y; f(y))dy = g(x). \quad (1.10)$$

Equations (1.4) and (1.6) in which $H(x, y, f) = K(x, y)f$, give rise to

linear Volterra integral equations of the form

$$\int_0^x K(x, y)f(y)dy = g(x) \quad (1.11)$$

and

$$f(x) - \int_0^x K(x, y)f(y)dy = g(x) \quad (1.12)$$

where $g(x)$ is termed the forcing function. (The term 'forcing function' is not always reserved for linear equations only.) Thus, any equations of the form

$$\int_0^x k(x - y)f(y)dy = g(x) \quad (1.13)$$

and

$$f(x) = \int_0^x k(x - y)f(y)dy + g(x) \quad (1.14)$$

are both linear and of convolution type of the first and second kind, respectively.

In our discussion of applications of Volterra integral equations we shall be particularly interested in (1.14) which gives rise to the classic renewal equation. The forcing function g is always continuous and the kernel $k(x)$ is at least of class L^1 (and hopefully continuous) on each finite sub-interval of $[0, \infty)$. Frequently, $k(x)$ is non-negative or non-positive. Often $g(x) \geq 0$ and $g(x)$ may even be monotone non-decreasing - cf Bellman & Cooke [7], Lotka [36], Feller [25].

We shall also be concerned with a special type of non-linear convolution equation in which the kernel H is such that

$$H(x, y, f) = k(x - y)\varphi(y, f) \quad (1.15)$$

1.3 Transformation of equations into new forms.

1.3.1 Volterra equations of first kind recast as equations of the second kind.

In general, solution of first kind equations is more difficult than solution of equations of the second kind. However, under certain conditions Volterra equations of the first kind may be transformed into equations of the second kind of Baker [4], Tricomi [50]. We shall illustrate by considering linear equations. Consider the linear Volterra equation of the first kind

$$\int_0^x K(x, y)f(y)dy = g(x) \quad (1.16)$$

where $K(x, y)$, $g(x)$ are continuously differentiable. We seek a continuous solution and require, therefore, that $g(0) = 0$.

Given that $\frac{\partial}{\partial x} K(x, y) = K_x(x, y)$ is continuous for $y \leq x$ and that $g'(x)$ is continuous, we obtain, on differentiating (1.16):

$$K(x, x)f(x) + \int_0^x K_x(x, y)f(y)dy = g'(x).$$

If $K(x, x) \neq 0$, then

$$f(x) + \int_0^x K_1(x, y)f(y)dy = \frac{g'(x)}{K(x, x)},$$

where $K_1(x, y) = \frac{K_x(x, y)}{K(x, x)}$. Thus, we obtain a Volterra equation of second kind.

If, on the other hand, $\frac{\partial}{\partial y} K(x, y) = K_y(x, y)$ is continuous for $y \leq x$, we may write

$$F(x) = \int_0^x f(y)dy$$

which is a Volterra equation of the first kind for $f(x)$, with solution $f(x) = F'(x)$. A Volterra equation of the second kind may be derived for $F(x)$ from (1.16) by using integration by parts for the left-hand term

to obtain

$$K(x, x)F(x) - \int_0^x K_y(x, y)F(y)dy = g(x).$$

Provided $K(x, x) \neq 0$ this results in

$$F(x) - \int_0^x K_2(x, y)F(y)dy = \frac{g(x)}{K(x, x)}$$

where $K_2(x, y) = \frac{K_y(x, y)}{K(x, x)}.$

The function $f(x)$ is then obtained by differentiating the solution $F(x)$ of this equation, Baker [4, p.9].

1.3.2. Linear Abel equations of the first kind recast as Volterra equations of the second kind.

Consider the linear Abel equation of the first kind

$$\int_0^x \frac{K(x, y)f(y)}{(x-y)^\alpha} dy = g(x). \quad (1.17)$$

It may be shown that, provided $K(x, x)$ and $g(x)$ satisfy certain conditions, equation (1.17) yields a Volterra equation of the second kind

$$f(x) + \int_0^x K^\#(x, y)f(y)dy = g^\#(x)$$

where

$$K^\#(x, y) = \frac{\sin \alpha\pi}{\pi} \int_0^1 \left\{ \frac{z}{1-z} \right\}^{1-\alpha} K_X(z(x-y)+y, y) dz$$

(cf. (1.7.3)) and

$$g^\#(x) = \frac{\sin \alpha\pi}{\pi} \frac{d}{dx} \int_0^x g(y)(x-y)^{\alpha-1} dy,$$

cf. Baker [4].

1.3.3. Other integral equations which may be recast as Volterra equations or as systems of Volterra equations.

We have noted that equations of the first kind may sometimes be transformed into equations of the second kind and that this is well known for equations of Volterra type, Tricomi [50]. Other integral equations

$$f(x) - \int_{\varphi_0(x)}^{\varphi_1(x)} K(x, y)f(y)dy = g(x)$$

and

$$\int_{\varphi_0(x)}^{\varphi_1(x)} K(x, y)f(y)dy = g(x),$$

where $\varphi_i(x)$ is continuous for $i = 0, 1$ and $K(x, y)$ is continuous for $\varphi_0(x) \leq y \leq \varphi_1(x)$, are sometimes similar to Volterra equations. In special circumstances, they may be recast as Volterra equations or as systems of Volterra equations. For example, consider the equation

$$f(x) - \int_{-x}^x K(x, y)f(y)dy = g(x) \quad (-\infty < x < \infty) \quad (1.10)$$

Suppose that $\varphi(x) = f(x)$ for $x \geq 0$, $\varphi(x) = 0$ for $x < 0$

and $\psi(x) = f(-x)$ for $x \geq 0$, $\psi(x) = 0$ for $x < 0$.

Then, for $x \geq 0$,

$$\varphi(x) - \int_0^x K(x, y)\varphi(y)dy - \int_0^x K(x, -y)\psi(y)dy = g(x)$$

and

$$\psi(x) + \int_0^x K(-x, -y)\psi(y)dy + \int_0^x K(-x, y)\varphi(y)dy = g(-x).$$

This is a system of coupled Volterra equations in which, if we determine $\varphi(x)$ and $\psi(x)$, we may set $f(x) = \varphi(x)$, $x \geq 0$ and $f(x) = \psi(-x)$, $x < 0$.

Thus the foregoing remarks in §§1.3.1/2/3 illustrate that Volterra

equations of the second kind are significant in the analysis of classical equations and justify the attention which we shall give later to equation (1.6). The direct treatment of other types of equation discussed above presents additional interesting problems.

1.4 Relations between integral equations and ordinary differential equations.

Many integral equations arise directly in the modelling of problems. However, they can also be deduced from differential equations. In these latter cases the integral equations are alternative formulations of problems whose initial mathematical formulation led to differential equations. Consider the initial value problem for a system of ordinary differential equations.

$$\tilde{f}'(x) = \tilde{F}(x, \tilde{f}(x)), \quad x \geq 0, \quad \tilde{f}(0) = \tilde{f}_0 \quad (1.20)$$

On integration, this yields

$$\tilde{f}(x) = \int_0^x \tilde{F}(y, \tilde{f}(y)) dy + \tilde{f}_0 \quad (1.21)$$

which is a vector-valued form of (1.6). (See e.g. Tricomi [50], Miller [41], Corduneanu [18], Barucha-Reid [6].)

Having observed this relationship between integral equations and ordinary differential equations we may mention here other significant results; see also Burton [13].

It is readily seen that the non-linear Volterra equation, with separable kernels, of the form

$$f(x) - \int_0^x \sum_{i=1}^N X_i(x) Y_i(y, f(y)) dy = g(x) \quad (1.22)$$

may be solved by writing

$$f(x) = g(x) + \sum_{i=1}^N a_i(x) X_i(x) \quad (1.23)$$

resulting in the functions $a_i(x)$ satisfying an initial value problem for a system of ordinary differential equations, Baker [4]. There also

exists the possibility of a similar development for certain convolution equations; Golberg [27]. Furthermore, we observe that the second kind Volterra equation with polynomial convolution kernel

$k(x - y) = \sum_{s=0}^S \lambda_s (x - y)^s$ may be reduced to a system of differential equations; Amini et al [1].

Indeed, large classes of Volterra equations may be reduced to linear differential equations with constant coefficients, the asymptotic stability of which can be completely determined by the Routh-Hurwitz criteria. We shall return to this point later.

Another tool in (for example) the analysis of numerical methods for second kind Volterra equations is the imbedding of the integral equation in a differential equation containing a parameter; see Wolkenfelt, van der Houwen & Baker [55]. The definition of the kernel function $H(x, y, f)$ of (1.6) is extended for $y > x$ and we define

$$\psi(t, x) = g(x) + \int_0^t H(x, y, f(y)) dy, \quad 0 \leq t \leq X, \quad (1.24)$$

where $f(x)$ satisfies the integral equation (1.6).

Clearly,

$$f(x) = \psi(x, x),$$

so that we may write (1.24) as

$$\psi(t, x) = g(x) + \int_0^t H(x, y, \psi(y, y)) dy. \quad (1.25)$$

On differentiating (1.25) with respect to t we obtain the initial value problem (Pouzet [46])

$$\begin{aligned} \frac{\partial}{\partial t} \psi(t, x) &= H(x, t, \psi(t, t)), \quad 0 \leq t, x \leq X, \quad (1.26) \\ \psi(0, x) &= g(x). \end{aligned}$$

This latter equation (1.26) may be regarded as a partial differential equation although, in the study of numerical methods, it has proved to be more convenient to treat it as an ordinary differential equation with x as parameter and t the independent variable; see Wolkenfelt [53], [54];

Wolkenfelt, van der Houwen & Baker [55].

We have commented earlier that in many cases integral equations are alternative formulations of problems which were initially modelled by systems of differential equations. Here, we indicate briefly a problem which may be formulated as a system of integral equations which, under certain circumstances, may have an alternative formulation as a system of ordinary differential equations. We refer to work of Hethcote & Tudor [29] who investigated models for endemic infectious diseases (i.e. diseases present for many years). They showed that those diseases for which infection confers permanent immunity may be modelled by a system of two non-linear Volterra integral equations of convolution type. For the non-cyclic SIR model (susceptible, infectious, removed) with vital dynamics (births and deaths) and immunization, the system reduces to a system of ordinary differential equations near the equilibrium point and to a system of delay-differential equations elsewhere. Thus, if one is interested in the behaviour near equilibrium points, or in the minimization necessary to cause the disease to die out, or in how the equilibrium points change as the parameter values and immunization rates change, then the ordinary differential equation models are sufficient. The authors of [29] employ Volterra equation theory to deduce stability results, and the paper illustrates, inter alia, the close relationship between differential, delay-differential and integral equations.

1.5 Integro-Differential Equations

1.5.1 As a member of the class of Volterra functional differential equations.

It is also our intention to consider an important type of Volterra equation, namely the Volterra integro-differential equation

$$f'(x) = G(x, f(x), \int_0^x H(x, y, f(y))dy), \quad x \geq 0 \quad (1.27a)$$

with prescribed initial condition

$$f(0) = f_0 \quad (1.27b)$$

Equations of this type have practical interest and arose, for example, in the work of Levin & Nohel [33] on reactor dynamics and may be found in predator-prey systems cf. Brauer [10], Cushing [20].

Equation (1.27) is a member of a class of equations called Volterra functional differential equations which have the form

$$f'(x) = G(x, f(\cdot)) \quad (1.28)$$

in which the value of $G(x, f(\cdot)) \equiv G(x, f)$, at a point depends on x and $f(t)$ for $0 \leq t \leq x$. This class also includes the delay-differential equations and, indeed, equation (1.27) is sometimes referred to as an infinite-delay equation since, in general, $f'(x)$ depends on all past values of $f(x)$. Thus, a number of papers on delay-differential equations which deal with the general equation (1.28) may also apply to equation (1.27a).

1.5.2 Relations between Volterra equations of second kind and integro-differential equations.

We may rewrite equation (1.27) to obtain the system

$$\begin{aligned} f_1(x) &= \int_0^x G(y, f_1(y), f_2(y)) dy + f_0 \\ f_2(x) &= \int_0^x H(x, y, f_1(y)) dy \end{aligned} \quad (1.29)$$

which is a coupled pair of integral equations. Using vector notation $\underline{\varphi}(x) = [f_1(x), f_2(x)]^T$ etc. we obtain the form

$$\underline{\varphi}(x) = \int_0^x \underline{\Phi}(x, s, \underline{\varphi}(s)) ds + \underline{\varphi}(0) \quad (1.30)$$

Thus, the integro-differential equation is written as a special case of a Volterra equation of the second kind and, hence, techniques for the numerical solution of Volterra integral equations may also be adapted to treat the system (1.30). However, this may be a convenient

arrangement rather than a practical one as a wider range of techniques may be obtained if each equation in (1.29) is treated separately.

On the other hand, just as differentiation of a Volterra equation of the first kind can, under certain conditions, produce a Volterra equation of the second kind so differentiation of (1.6), assuming that f, g, H satisfy appropriate conditions, results in a form of (1.27). Thus, assuming the necessary conditions are satisfied, on differentiating (1.6) we obtain

$$f'(x) - H(x, x, f(x)) - \int_0^x H_x(x, y, f(y))dy = g'(x),$$

where $H_x(x, y, f(y)) = \frac{\partial}{\partial x} H(x, y, f(y)).$

On re-arranging, this results in a form of (1.27) as follows:

$$f'(x) = g'(x) + H(x, x, f(x)) + \int_0^x H_x(x, y, f(y))dy.$$

1.6 Variation of Constants Formula.

In the study of stability and asymptotic behaviour of solutions of ordinary differential equations, use can be made of the variation-of-constants formulae. The effect of perturbations in (1.6) and (1.27) upon the solution $f(x)$ is in theory obtainable by analysing variation-of-constants formulae; see Brunner [11]. However, the general variation-of-constants theory for integral and integro-differential equations is not simple except in the case of particular kernels.

1.6.1 The Volterra integral equation of the second kind.

Consider the linear case of the Volterra equation of the second kind

$$f(x) - \int_0^x K(x, y)f(y)dy = g(x). \quad (1.31a)$$

We may define $R(x, y)$ the resolvent kernel, associated with the kernel $K(x, y)$, satisfying

$$R(x, y) - \int_y^x K(x, u)R(u, y)du = K(x, y) \quad (1.31b)$$

and express the solution of (1.31a) in terms of g as

$$f(x) = g(x) + \int_0^x R(x, u)g(u)du. \quad (1.31c)$$

This is, in essence, the variation-of-constants formula. Brunner [11].

To verify (1.31c) multiply (1.31a) on the left by $R(x, u)$ and integrate from 0 to x :

$$\int_0^x R(x, u) f(u) du - \int_0^x R(x, u) \int_0^u K(u, y) f(y) dy du = \int_0^x R(x, u) g(u) du.$$

Thus

$$\int_0^x R(x, u) f(u) du - \int_0^x \int_y^x R(x, u) K(u, y) du f(y) dy = \int_0^x R(x, u) g(u) du. \quad (1.32)$$

Miller [40, p200] shows that

$$\int_y^x R(x, u) K(u, y) du = \int_y^x K(x, u) R(u, y) du$$

and thus (1.31b) may be written

$$R(x, y) - \int_y^x R(x, u) K(u, y) du = K(x, y) \quad (1.33)$$

Hence (1.32), together with (1.33), yields

$$\int_0^x R(x, u) f(u) du - \int_0^x [R(x, y) - K(x, y)] f(y) dy = \int_0^x R(x, u) g(u) du$$

Thus

$$\int_0^x K(x, y) f(y) dy = \int_0^x R(x, u) g(u) du$$

and together with (1.31a) yields (1.31c). The above manipulation is more transparent if we use operator notation (see §1.8.2).

Further, the solution $U(x, y)$ of the equation

$$U(x, y) = 1 + \int_y^x K(x, z)U(z, y)dz \quad (1.34)$$

is

$$U(x, y) = 1 + \int_y^x R(x, z)dz. \quad (1.35)$$

We could define $R(x, y)$ by

$$R(x, y) = - \frac{\partial U}{\partial y}(x, y) \quad (1.36)$$

to show that the resolvent kernel $R(x, y)$ is related to the function $U(x, y)$, which is known as the differential resolvent. Now, from (1.31c) and (1.36) we have

$$f(x) = g(x) - \int_0^x \frac{\partial U}{\partial z}(x, z)g(z)dz.$$

On integrating by parts this yields

$$f(x) = U(x, 0)g(0) + \int_0^x U(x, y)g'(y)dy. \quad (1.37)$$

Hence, to solve the integral equation (1.31a), either the function $R(x, y)$ or $U(x, y)$ would enable us to find $f(x)$ using either (1.31c) or (1.37) respectively.

In classical texts the resolvent kernel is developed from the Neumann series: $R(x, y) = \sum_{j=1}^{\infty} K^j(x, y)$, where

$$K^j(x, y) = \int_0^x K(x, z)K^{j-1}(x, y)dz. \text{ In the case when } K(x, y) = k(x - y)$$

is of convolution type then $R(x, y) = r(x - y)$ is also of convolution type. When the kernel is constant that is, $K(x, y) = K$, then

$$r(x - y) - \int_y^x K r(u - y) du = K.$$

Here, the variable y occurs in the equation only as a parameter, if the role of the parameter is suppressed

$$r(x) - \int_0^x K r(u) du = K$$

where

$$r(x - 0) \equiv r(x).$$

This is equivalent to a linear differential equation and it follows that $r(x) = Ke^{Kx}$. The above formulae for the resolvent $R(x, y)$ are also applicable to Abel equations of the second kind. They illustrate the important fact, alluded to in 1.2.2 that the solution of (1.5) may be expected to have a weak form of singularity at $x = 0$ when $g(x)$ is well-behaved.

1.6.2 The integro-differential equation.

Brauer [9] has developed an analogous variation-of-constants theory for the linear integro-differential equation

$$f'(x) = c(x)f(x) + \int_0^x K^\#(x, s)f(s)ds, \quad x \geq 0, \quad f(0) = f_0 \quad (1.38)$$

which is equivalent to the integral equation of the form (1.31a) with $g(x) \equiv f_0$ and

$$K(x, y) = c(y) + \int_y^x K^\#(u, y) du \quad (1.39)$$

The equations, for the resolvent kernel $R(x, y)$ and the differential resolvent $U(x, y)$, are, respectively,

$$\frac{\partial R}{\partial x}(x, y) = c(x)R(x, y) + \int_y^x K^\#(x, u)R(u, y)du + K^\#(x, y), \quad R(x, x) = c(x), \quad (1.40)$$

and

$$\frac{\partial U}{\partial x}(x, y) = c(x)U(x, y) + \int_y^x K^\#(x, z)U(z, y)dz, \quad U(x, x) = 1. \quad (1.41)$$

Their solutions $R(x, y)$ and $U(x, y)$ are given by (1.31b) and (1.35).

Thus, the solution $f(x)$ of (1.38) is given by

$$f(x) = f_0 + \int_0^x R(x, y)f_0 dy = U(x, 0)f_0. \quad (1.42)$$

Hence, to solve the given integro-differential equation we need to solve (1.40) or (1.41) which may be worse than the given problem.

Turning to the non-linear systems of Volterra equations we find that the resolvent kernel can also be used to write some of the systems in a form which corresponds to the variation-of-constants formula for ordinary differential equations.

Consider the non-linear system

$$f(x) = g(x) + \int_0^x K(x, s)[f(s) + k(s, f(s))]ds \quad (1.43)$$

where $k(x, y)$ represents "small terms" or terms of "higher order" in x . If the solution $f(x)$ is assumed to be known then (1.43) can be written in the form

$$f(x) = G(x) + \int_0^x K(x, s)f(s)ds \quad (1.44)$$

where

$$G(x) = g(x) + \int_0^x K(x, s)k(s, f(s))ds. \quad (1.45)$$

From these latter equations it can be shown that the solution $f(x)$ of (1.43) is given by

$$f(x) = F(x) + \int_0^x R(x, s)k(s, f(s))ds \quad (1.46)$$

where $F(x)$ is the solution of the linear integral equation (1.31a)

and $R(x, y)$ is the resolvent kernel given by (1.31b). Now (1.46) is a nonlinear equation for f whose solution can be investigated via the natural iteration (say).

$$f_{n+1}(x) = F(x) + \int_0^x R(x, s) k(s, f_n(s)) ds \quad (1.47)$$

1.7 The role of Laplace Transforms.

1.7.1 Integral equations of the second kind and of convolution type.

We return to a linear system of Volterra equations of convolution type having the form

$$f(x) - \int_0^x k(x - y) f(y) dy = g(x) \quad (1.48)$$

and we find a role for Laplace transforms Doetsch [23], Watson [52].

The Laplace transform of f where $f: [0, \infty) \rightarrow \mathbb{R}$ is defined as

$$L(f) = F(s) = \int_0^\infty e^{-st} f(t) dt. \quad (1.49)$$

We know that, under certain conditions,

$$L\left[\int_0^x k(x - y) f(y) dy\right] = L(k)L(f). \quad (1.50)$$

Hence, taking the Laplace transforms (assuming that they exist) of (1.48) we have

$$L(f) - L(k)L(f) = L(g)$$

and

$$L(f) = \frac{L(g)}{1 - L(k)} \quad (1.51)$$

Comparing (1.48) and (1.51) we see that our integral equation with an arbitrary convolution kernel $k(x - y)$ has been transformed into the problem of inverting a Laplace transform, that is, find $f(x)$ such that

$$\int_0^\infty e^{-xs} f(x) dx = F(s) = \frac{G(s)}{1 - K(s)}. \quad (1.52)$$

For this last equation, provided suitable conditions exist [52], we have the inversion formula

$$f(x) = L^{-1}[F(s)] = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{xs} F(s) ds \quad (1.53)$$

where σ is greater than the real part of all the singularities of $F(s)$ (cf Watson [52], Doetsch [23]).

1.7.2 Relations with the Resolvent.

If the resolvent of (1.48) is $r(x - y)$ then

$$f(x) = g(x) + \int_0^x r(x - s)g(s)ds \quad (1.54)$$

Hence,

$$L(f) = L(g) + L(r)L(g) \quad (1.55)$$

Thus, from (1.51) and (1.55)

$$L(r) = -1 + \frac{L(f)}{L(g)} = -1 + \frac{1}{1 - L(k)} = \frac{L(k)}{1 - L(k)}$$

Hence

$$r(x) = L^{-1} \left[\frac{K(s)}{1 - K(s)} \right] \quad (1.56)$$

1.7.3 Integral equations of first kind and of convolution type.

Consider the linear integral equation of the first kind and of convolution type

$$\int_0^x k(x - y)f(y)dy = g(x) \quad (1.57)$$

The immediate application of the Laplace transformation yields

$$K(s)F(s) = G(s) \quad (1.58)$$

that is,

$$F(s) = \frac{G(s)}{K(s)} \quad (1.59)$$

However, $[K(s)]^{-1}$ cannot be a Laplace transform as $[K(s)]^{-1} \rightarrow \infty$ as

$s \rightarrow \infty$ and, without conditions on g , there are no results analogous to those of §1.7.1/2.

Sometimes, as illustrated in §1.3.1, the integral equation of the first kind can be converted into one of the second kind which can then be handled as in 1.7.1. Thus, if $k(x)$ and $g(x)$ are continuously differentiable and $k(0) \neq 0$, by differentiation of (1.57), we obtain an integral equation of the second kind

$$k(0)f(x) + \int_0^x k'(x-y)f(y)dy = g'(x).$$

If, further, $k(0) = k'(0) = \dots = k^{n-1}(0) = 0$, $k^n(0) \neq 0$ then, by differentiating (1.57) $n+1$ times, we obtain

$$k^n(0)f(x) + \int_0^x k^{n+1}(x-y)f(y)dy = g^{n+1}(x).$$

Again, this is an integral equation of the second kind.

However, this method fails if $k(x)$ possesses no derivatives at $x = 0$. For example, $k(x) = x^{-\alpha}$, $0 < \alpha < 1$, is such a function. In these cases, the following method may lead to the desired result:

In place of the function $f(x)$ we consider the function

$$\varphi(x) = \int_0^x f(s)ds. \quad (1.60)$$

Thus, $L(\varphi) = \varphi(s) = \frac{1}{s} F(s)$

and instead of (1.59) we find

$$\Phi(s) = \frac{G(s)}{sK(s)}. \quad (1.61)$$

If $1/[sK(s)]$ is an L-transform then the inversion theorem will yield $\varphi(x)$ and $f(x)$ is found by differentiation of $\varphi(x)$.

As an example we may consider the Abel integral equation

$$\int_0^x \frac{f(y)}{(x-y)^\alpha} dy = g(x) \quad (0 < \alpha < 1) \quad (1.62)$$

which is of importance in many branches of physics. From the definition of the gamma function

$$\Gamma(\nu) = \int_0^{\infty} x^{\nu-1} e^{-x} dx, \quad \text{Re } \nu > 0 \quad (1.63)$$

we may show that

$$L(x^{-\alpha}) = \frac{\Gamma(1-\alpha)}{s^{1-\alpha}} = K(s) \quad \text{provided } \alpha < 1. \quad (1.64)$$

Thus, from (1.61)

$$\Phi(s) = \frac{G(s)}{\Gamma(1-\alpha)s^{\alpha}}. \quad (1.65)$$

It is also well known that

$$L^{-1}(s^{-\alpha}) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} \quad \text{provided } \alpha > 0. \quad (1.66)$$

Hence, for $0 < \alpha < 1$, the original function corresponding to $\Phi(s)$ is

$$\varphi(x) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^x y^{\alpha-1} g(x-y) dy. \quad (1.67)$$

Since

$$\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} = \frac{\sin \alpha\pi}{\pi}, \quad \text{then}$$

$$\varphi(x) = \frac{\sin \alpha\pi}{\pi} \int_0^x y^{\alpha-1} g(x-y) dy \quad (1.68)$$

If $g(x)$ is differentiable and continuous for $x = 0$, on differentiating (1.68), we obtain (see §1.1)

$$f(x) = \varphi'(x) = \frac{\sin \alpha\pi}{\pi} \left[g(0)x^{\alpha-1} + \int_0^x y^{\alpha-1} g'(x-y) dy \right]. \quad (1.69)$$

*1.8 Background Material

We pause to introduce some background material which is relevant to our later discussions. A more rigorous treatment may be found in Cheney [15], Smithies [48], Cochran [16], Mikhlin [38]. The sophisticated reader may proceed directly to §1.9.

1.8.1 Function Spaces

Some of the ideas in functional analysis are valuable in the theoretical and numerical solution of integral equations. Certain basic ideas of such analysis are introduced here but a more advanced treatment may be found, for example, in Collatz [17].

(a) Normed linear space

The set $C[a, b]$ of complex-valued functions which are continuous on $[a, b]$ form a linear space if, when φ_1 and φ_2 are members of $C[a, b]$, then

$$(\varphi_1 + \varphi_2)(x) = \varphi_1(x) + \varphi_2(x)$$

and $(\alpha \varphi_1)(x) = \alpha \varphi_1(x)$, α constant.

A norm of $\varphi(x)$ denoted $\|\varphi(x)\|$ is required to have the properties

(i) $\|\varphi(x)\| = 0$ if and only if $\varphi(x) = 0$,

(ii) $\|\alpha \varphi(x)\| = |\alpha| \|\varphi(x)\|$,

(iii) $\|(\varphi_1 + \varphi_2)(x)\| \leq \|\varphi_1(x)\| + \|\varphi_2(x)\|$.

Thus, a linear space with a defined norm is referred to as a normed linear space.

Example 1.1

We illustrate the concepts above.

(a) \mathbb{R}^n : $\|x\| = \|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$, $p \geq 1$.

$p = 2$ gives the Euclidean norm.

$p = \infty$ gives the Maximum or Chebyshev norm $\|x\|_\infty = \max |x_i|$.

(b) Real-valued functions on $[a, b]$ such that $\int_a^b |f(x)|^p dx < \infty$,
 $1 \leq p < \infty$ with Holder norm $\|f\|_p = \left\{ \int_a^b |f(x)|^p dx \right\}^{1/p}$, $\|f\|_\infty = \sup |f(x)|$.

(b) Convergence

Consider a normed linear space, say $C[a, b]$ with $\|\cdot\|$. When we have a sequence of functions $\{\varphi_n(x)\}$ ($n = 0, 1, \dots$) and $\|\varphi(x) - \varphi_n(x)\| \rightarrow 0$ as $n \rightarrow \infty$ we say that $\varphi_n(x)$ converges in norm to $\varphi(x)$. With the uniform norm $\|\varphi(x)\|_\infty = \sup_{a \leq x \leq b} |\varphi(x)|$ we speak of uniform convergence and if the norm is $\|\cdot\|_2$ where $\|\varphi(x)\|_2 = \left\{ \int_a^b |\varphi(x)|^2 dx \right\}^{1/2}$, we speak of convergence in the mean. In addition, we say that $\varphi_n(x)$ converges relatively uniformly to $\varphi(x) \in [a, b]$ if there is some $\psi(x)$, which is usually square-integrable, such that

$$\|\varphi_n(x) - \varphi(x)\| \leq \varepsilon_n \psi(x) \quad \text{for } a \leq x \leq b, \quad \text{where } \lim_{n \rightarrow \infty} \varepsilon_n = 0.$$

(c) A metric defined on a function space

We obtain a metric $d(\varphi_1, \varphi_2)$ on a function space when d satisfies the following properties:

- (i) $d(\varphi_1, \varphi_2) \geq 0$; (ii) $d(\varphi_1, \varphi_2) = 0 \Leftrightarrow \varphi_1(x) = \varphi_2(x)$;
- (iii) $d(\varphi_1, \varphi_2) = d(\varphi_2, \varphi_1)$; (iv) $d(\varphi_1, \varphi_3) \leq d(\varphi_1, \varphi_2) + d(\varphi_2, \varphi_3)$.

The natural (or induced) metric on a normed linear space is $d(\varphi_1, \varphi_2) = \|\varphi_1(x) - \varphi_2(x)\|$. When a metric $d(\cdot, \cdot)$ on a linear space satisfies (i)-(iv) we call the linear space a metric space.

(d) Cauchy sequence

A sequence $\{\varphi_n(x)\}$ in a space is a fundamental or Cauchy sequence if, given $\varepsilon > 0$, there exists $N(\varepsilon)$ such that

$$d(\varphi_n, \varphi_m) \leq \varepsilon, \quad \text{for all } n, m \geq N(\varepsilon).$$

If $d(\varphi, \varphi_n) \rightarrow 0$ as $n \rightarrow \infty$, we say that $\varphi_n(x)$ converges to $\varphi(x)$.

(e) Completeness

A space is said to be complete if every fundamental sequence converges to an element in the space. (A complete normed linear space is a Banach space.)

Example 1.2 (Cheney [15]) We illustrate the preceding concept.

$$\text{Define} \quad \|\varphi(x)\| = \sup_{a \leq x \leq b} |\varphi(x)|$$

The induced metric for the space $C[a, b]$ of continuous functions is $d(\varphi_1, \varphi_2) = \sup_{a \leq x \leq b} |\varphi_1(x) - \varphi_2(x)|$.

Then the space is complete and convergence with respect to the induced metric is described as uniform convergence.

(f) Closed sets

Suppose that S is a subset of a normed linear space X with a norm $\|\cdot\|$. $S \subset X$ is closed if every convergent sequence in S has its limit in S .

(g) Compactness

A subset S of a metric space X is said to be compact in X if every sequence of elements in S contains a subsequence which converges (with respect to the given metric) to an element of S . Thus, if S is closed and compact in X it is compact in itself.

Example 1.3 We illustrate compactness

Consider S to be the set of functions $\{\varphi(x)\}$ which are continuous and differentiable on $[a, b]$ with $\|\varphi(x)\|_\infty \leq 1$ and $\|\varphi'(x)\|_\infty \leq 1$. Then S is compact in $C[a, b]$, where the metric is that induced by $\|\cdot\|_\infty$.

(h) Equi-continuity

A set of functions S defined in $[a, b]$ is said to be equi-continuous if, given $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$ such that for any $x_1, x_2 \in [a, b]$ satisfying $|x_1 - x_2| < \delta$ we have $|f(x_1) - f(x_2)| < \epsilon$ for all $f(x) \in S$.

Example 1.4

$S := \{\varphi(x) : \|\varphi'(x)\|_\infty \leq 1; \varphi \in C^1[0, 1]\}$ is equi-continuous in $C[0, 1]$.

1.8.2 Linear Operators

X and Y are linear spaces and T is an operator such that $T : X \rightarrow Y$. T is called a linear operator if

$$T(\varphi_1 + \varphi_2) = T\varphi_1 + T\varphi_2, \quad \text{for all } \varphi_1, \varphi_2 \in X$$

$$T(\alpha\varphi) = \alpha T\varphi, \quad \text{for all } \varphi \in X, \alpha \in \mathbb{R}.$$

If X and Y are normed linear spaces, a linear operator T is bounded if there is an M such that

$$\|T\varphi\| \leq M\|\varphi\|, \quad \text{for all } \varphi \in X.$$

If we call the least value of M the norm of T then

$$\|T\| = \sup_{\substack{\varphi \in X \\ \varphi \neq 0}} \frac{\|T\varphi\|}{\|\varphi\|}.$$

Example 1.5

A bounded linear operator is continuous.

(Proof: T is bounded $\Rightarrow \|T\varphi_1 - T\varphi_2\| \leq \|T\| \|\varphi_1 - \varphi_2\| < \epsilon$ for all $\varphi_1, \varphi_2 \in X$ such that $\|\varphi_1 - \varphi_2\| < \frac{\epsilon}{\|T\|}$.)

An operator is continuous if it maps every convergent sequence into a corresponding convergent sequence. A continuous operator which maps a normed space S_1 into a normed space S_2 is said to be completely continuous if the image of any completely bounded set $B \subset S_1$, is compact in S_2 . It is sufficient, by linearity of T , to consider the 'unit ball' B of elements with norm ≤ 1 , $B = \{\varphi \in X: \|\varphi\| \leq 1\}$.

Linear Integral Operators

Associated with a kernel $K(x, y)$ and an interval of integration $[a, b]$ is a linear integral operator K which transforms a function φ into a new function ψ where $\psi = K\varphi$ and $\psi(x)$ is defined

$$\psi(x) = \int_a^b K(x, y)\varphi(y)dy.$$

We assume that $K(x, y)$ is at least piecewise continuous and hence $\psi(x)$ exists if $\varphi(x)$ is integrable: if the complex-valued $K(x, y)$ is continuous on $[a, b]$ then $(K\varphi)(x) \in C[a, b]$ when $\varphi(x)$ is continuous. Then the integral operator K 'maps' $C[a, b]$ onto itself and is linear as

$$K(\alpha_1\varphi_1 + \alpha_2\varphi_2)(x) = \alpha_1(K\varphi_1)(x) + \alpha_2(K\varphi_2)(x), \text{ for } \alpha_1, \alpha_2 \text{ complex values.}$$

Suppose $K(x, y) = 0$ for $y > x$; then (taking $a = 0$, $b = X$) the linear equation

$$f(x) - \lambda \int_0^x K(x, y)f(y)dy = g(x) \quad (0 \leq x \leq X) \quad (1.70)$$

may be represented as

$$(I - \lambda K)f = g. \quad (1.71)$$

The determination of the resolvent, referred to in §1.6.1, may be regarded as analogous to finding the inverse of a matrix. Thus, if the solution of (1.70) is given by (see equation (1.31c))

$$f(x) = g(x) + \lambda \int_0^x R(x, u)g(u)du \quad (1.72)$$

(R depends on λ); this may also be represented in operator notation as

$$f = (I + \lambda R)g; \quad R \equiv R_\lambda. \quad (1.73)$$

Hence, $(I - \lambda K)(I + \lambda R) = I$ and we see that the inverse operator of $I - \lambda K$ is $I + \lambda R$. Further, $\lambda \neq 0$, $R - K = \lambda KR$.

Thus $R(x, z) - \lambda \int_y^x K(x, y)R(y, z)dy = K(x, z)$ which corresponds to (1.31b). Just as one does not normally compute the inverse of a matrix to solve a system of linear equations, finding $R(x, y)$ for any given x can be as difficult or more difficult, than finding $f(x)$. However, knowledge of the qualitative behaviour of

R can give valuable insight.

The Arzela-Ascoli Theorem 1.1 (Courant & Hilbert [19])

This states that a necessary and sufficient condition that a set of functions which are continuous on $[a, b]$ should be compact on $[a, b]$ (with uniform norm) is that the set should be (i) uniformly bounded and (ii) equi-continuous.

Example 1.6

The above theorem permits us to investigate whether or not an integral operator is completely continuous on the space $C[a, b]$. Consider the operator K ,

$$K\varphi(x) = \int_a^b K(x, y)\varphi(y)dy,$$

where $\lim \int_a^b |K(x', y) - K(x'', y)| dy \rightarrow 0$ as $|x' - x''| \rightarrow 0$ and

where $\sup_{a \leq x \leq b} \int_a^b |K(x, y)| dy < \infty$ and $\|\varphi(y)\|_\infty \leq 1$. Write $\psi(x) = K\varphi(x)$

and note that $\|K\|_\infty < \infty$, so that K is a bounded linear operator. Then

$$\|\psi(x)\|_\infty = \|K\varphi(x)\|_\infty \leq \|K\|_\infty \|\varphi(x)\|_\infty \leq \|K\|_\infty.$$

Thus $\{\psi(x)\}$ are uniformly bounded.

For equi-continuity consider δ, ϵ such that

$$|\psi(x_1) - \psi(x_2)| = \left| \int_a^b K(x_1, y)\varphi(y)dy - \int_a^b K(x_2, y)\varphi(y)dy \right|$$

$$\leq \int_a^b |K(x_1, y) - K(x_2, y)| |\varphi(y)| dy$$

$$\leq \int_a^b |K(x_1, y) - K(x_2, y)| dy, \text{ given } \|\varphi(x)\|_\infty \leq 1,$$

$$< \epsilon \text{ when } |x_1 - x_2| < \delta.$$

by the uniform continuity of $K(x, y)$ on $[a, b]^2$.

This holds uniformly for every ψ under consideration, thus $\{\psi(x)\}$ are equi-continuous. Hence by the Arzela-Ascoli Theorem $\{\psi(x)\}$ are compact

on $[a, b]$ and it follows that the integral operator K is completely continuous on $C[a, b]$. It is readily seen that when $K(x, y) = 0$ for $y > x$, and $K(x, y)$ is continuous for $a \leq y \leq x \leq b$, the assumptions are satisfied so a continuous Volterra kernel defines a compact operator on $C[a, b]$.

Contraction Operators

Let (X, d) be a metric space and $T: X \rightarrow X$. The operator T is a contraction operator if there is an $\alpha \in [0, 1)$ such that $\varphi, \psi \in X$ imply

$$d[T(\varphi), T(\psi)] \leq \alpha d(\varphi, \psi).$$

Theorem 1.2 (Contraction Mapping Principle)

Let (X, d) be a complete metric space and $T: X \rightarrow X$ a continuous contraction operator. Then there is a unique $\varphi \in X$ with $T(\varphi) = \varphi$. Furthermore if $\psi \in X$ and if $\{\psi_n\}$ is defined inductively by $\psi_1 = T(\psi_0)$ and $\psi_{n+1} = T(\psi_n)$ then $\psi_n \rightarrow \varphi$, the unique fixed point. That is, the equation $T(\varphi) = \varphi$ has one and only one solution.

Proof

Let $\varphi_0 \in X$ and define a sequence $\{\varphi_n\}$ in X by $\varphi_1 = T(\varphi_0)$, $\varphi_2 = T(\varphi_1) = T^2(\varphi_0)$, ..., $\varphi_n = T(\varphi_{n-1}) = T^n(\varphi_0)$. To see that $\{\varphi_n\}$ is a Cauchy sequence, for $m > n$ we have

$$\begin{aligned} d(\varphi_n, \varphi_m) &= d(T^n \varphi_0, T^m \varphi_0) \\ &\leq \alpha d(T^{n-1} \varphi_0, T^{m-1} \varphi_0) \\ &\vdots \\ &\leq \alpha^n d(\varphi_0, \varphi_{m-n}) \\ &\leq \alpha^n \{d(\varphi_0, \varphi_1) + d(\varphi_1, \varphi_2) + \dots + d(\varphi_{m-n-1}, \varphi_{m-n})\} \\ &\leq \alpha^n \{d(\varphi_0, \varphi_1) + \alpha d(\varphi_0, \varphi_1) + \dots + \alpha^{m-n-1} d(\varphi_0, \varphi_1)\} \\ &= \alpha^n d(\varphi_0, \varphi_1) \{1 + \alpha + \dots + \alpha^{m-n-1}\} \\ &= \alpha^n d(\varphi_0, \varphi_1) \left\{ \frac{1 - \alpha^{m-n}}{1 - \alpha} \right\}. \end{aligned}$$

Since $\alpha < 1$, the right hand side approaches zero as $n \rightarrow \infty$. Hence $\{\varphi_n\}$ is a Cauchy sequence and, because (X, d) is complete, it has a limit $\varphi \in X$. T is continuous so

$$T(\varphi) = T(\lim_{n \rightarrow \infty} \varphi_n) = \lim_{n \rightarrow \infty} T(\varphi_n) = \lim_{n \rightarrow \infty} \varphi_{n+1} = \varphi$$

and φ is a fixed point.

To demonstrate uniqueness let $T(\varphi) = \varphi$ and $T(\psi) = \psi$.

Then $d(\varphi, \psi) = d(T(\varphi), T(\psi)) \leq \alpha d(\varphi, \psi)$ and, as $\alpha < 1$, $d(\varphi, \psi) = 0$ which implies that $\varphi = \psi$.

Various forms of fixed point theorem can be found in the literature. For later purposes we shall use the following (see Collatz [17, p.357]) which differs from the foregoing result in some details:

Theorem 1.3 (Schauder's fixed point theorem)

Let S be a closed bounded convex set in a Banach space $(X, \|\cdot\|)$ and let T be a completely continuous mapping of S into itself. Then there exists at least one point $\varphi \in S$ such that $\varphi = T(\varphi)$. [Proof: see Collatz [17].]

1.9 Topics in the Theory of Volterra Equations

In this section we consider, briefly, some of the questions regarding the behaviour of solutions of Volterra equations. Problems of particular interest concern questions of existence and uniqueness, boundedness, behaviour of solutions as the independent variable approaches infinity and existence of periodic solutions. The qualitative theory for integral equations has been developed less than the qualitative theory which exists for ordinary differential equations. In consequence, many of the problems outlined above are discussed in relation to integral equations which have special forms. We shall frequently refer to a useful survey of problems in qualitative

behaviour of solutions of Volterra equations which is to be found in Tsalyuk [51]. There is also a useful survey due to Nohel [44] which, although not exhaustive, deals with some equations which are of considerable mathematical interest and which originate from investigations in physics and engineering. More recently, Burton [13] has attempted to develop a systematic treatment of the theory of Volterra integral and integro-differential equations so that whenever possible it parallels the qualitative theory for ordinary differential equations.

1.9.1 Existence and Uniqueness

In this section we shall indicate elementary proofs of the existence and uniqueness of solutions of Volterra equations. We illustrate by using both iterative and fixed point methods cf. Wouk [56], Thielman [49]. Additionally, for the linear equation, we may refer for example, to Cochran [16, pp.64 et seq.], Burton [13, pp.24-25]. However, before proceeding to the proof of the uniqueness of the solution of certain Volterra equations we pause to introduce a result which will facilitate the proof of the theorem.

LEMMA (Gronwall's Inequality - cf. Burton [13, p.24])

Let $f, g : [0, X] \rightarrow [0, \infty)$ be continuous and let c be a non-negative number. If

$$f(x) \leq c + \int_0^x g(y)f(y)dy, \quad x \in [0, X]$$

then

$$f(x) \leq c \exp\left\{\int_0^x g(y)dy\right\}, \quad x \in [0, X].$$

Proof Suppose $c > 0$. Divide the given relation by $c + \int_0^x g(y)f(y)dy$ and multiply by $g(x)$ to obtain

$$\frac{f(x)g(x)}{c + \int_0^x g(y)f(y)dy} \leq g(x).$$

Integration from 0 to x yields

$$\ln\{[c + \int_0^x g(y)f(y)dy]/c\} \leq \int_0^x g(y)dy$$

Hence, $f(x) \leq c + \int_0^x g(y)f(y)dy \leq c \exp \int_0^x g(y)dy.$

If $c = 0$; take the limit as $c \rightarrow 0$ through positive values.

Theorem 1.4

Consider the linear equation

$$f(x) = g(x) + \int_0^x K(x, y)f(y)dy, \quad x \in [0, X] \quad (1.74)$$

in which $g(x)$ and $K(x, y)$ are continuous for $0 \leq y \leq x \leq X$ and

$$L = \max_{0 \leq y \leq x \leq X} |K(x, y)|, \quad M = \max_{0 \leq x \leq X} |g(x)|, \quad \text{then there is one}$$

and only one solution $f(x)$ of (1.74) on $[0, X]$.

Proof

The procedure involves a sequence of functions $\{f_n(x)\}$ on $[0, X]$ defined as Picard's successive approximations.

$$\begin{aligned} f_0(x) &= g(x) \\ f_{\nu+1}(x) &= g(x) + \int_0^x K(x, y)f_{\nu}(y)dy, \quad \nu \geq 0. \end{aligned} \quad (1.75)$$

From (1.75) we have, by induction,

$$|f_{\nu+1}(x) - f_{\nu}(x)| = \left| \int_0^x [K(x, y)f_{\nu}(y) - K(x, y)f_{\nu-1}(y)]dy \right|$$

$$\leq \int_0^x |K(x, y)| |f_\nu(y) - f_{\nu-1}(y)| dy$$

$$\leq L \int_0^x |f_\nu(y) - f_{\nu-1}(y)| dy$$

$$\leq \frac{ML^{\nu+1}x^{\nu+1}}{(\nu+1)!},$$

$$\leq \frac{ML^{\nu+1}x^{\nu+1}}{(\nu+1)!}, \quad x \in [0, X].$$

By comparison with the exponential series, we see that the f_n form a Cauchy sequence in the space of continuous functions on $[0, X]$. Thus, it follows from the Arzela-Ascoli Theorem that there exists a uniformly convergent subsequence and hence a continuous limit function $F(x)$, say, which is a solution of (1.74).

To see that $F(x)$ is the only solution, suppose that there are two solutions, say $F_1(x)$ and $F_2(x)$ on an interval $[0, X]$. From (1.74)

$$F_1(x) - F_2(x) = \int_0^x K(x, y) \{F_1(y) - F_2(y)\} dy$$

so that $|F_1(x) - F_2(x)| \leq L \int_0^x |F_1(y) - F_2(y)| dy.$

This relation is of the form

$$G(x) \leq c + \int_0^x L G(y) dy$$

with $c = 0$. By Gronwall's inequality $G(x) \leq c e^{Lx} = 0$.

Hence $F_1(x) = F_2(x)$.

Similar procedures to those presented above are applicable to general non-linear Volterra equations, Tricomi [50, pp.42 et seq.], Cochran [16, pp.68 et seq.], Nohel [43]. However, to illustrate the

fixed point method we use the Contraction Mapping Principle to demonstrate the existence and uniqueness of the solution of certain non-linear Volterra equations; cf. Burton [13, pp.73-74].

Consider the general non-linear Volterra equation

$$\tilde{f}(x) = \tilde{g}(x) + \int_0^x \tilde{H}(x, y, \tilde{f}(y)) dy. \quad (1.76)$$

We recall from §1.5.2 that an integro-differential equation with initial conditions can be put in the form (1.76). Our next theorem is similar to that given by Kershaw in Baker and Phillips [30], see also Saaty [47].

Theorem 1.5

Let a^* , b^* and L be positive numbers and take some fixed $\alpha \in (0, 1)$. Suppose

- (i) \tilde{g} is continuous on $[0, a^*]$,
- (ii) \tilde{H} is continuous on $B = \{(x, y, f) : 0 \leq y \leq x \leq X \text{ and } \|f - \tilde{g}\| \leq b^*\}$,
- (iii) \tilde{H} satisfies a Lipschitz condition with respect to f on B such that

$$\|\tilde{H}(x, y, \tilde{f}_1) - \tilde{H}(x, y, \tilde{f}_2)\| \leq L\|\tilde{f}_1 - \tilde{f}_2\|$$

if $(x, y, \tilde{f}_1), (x, y, \tilde{f}_2) \in B$.

If $M = \sup \|\tilde{H}(x, y, \tilde{f})\|$ then there is a solution of (1.76) on $[0, X]$ where $X = \min[a^*, \frac{b^*}{M}, \frac{\alpha}{L}]$.

Proof

Let S be the set of continuous functions from $[0, X] \rightarrow \mathbb{R}^n$ with $\tilde{\psi} \in S$ if

$$\|\tilde{\psi} - \tilde{g}\|_\infty := \max_{0 \leq x \leq X} \|\tilde{\psi}(x) - \tilde{g}(x)\| \leq b^*.$$

S is a convex neighbourhood of \tilde{g} .

Define a completely continuous operator T on S by

$$T(\underline{\psi})(x) = \underline{g}(x) + \int_0^x \underline{H}(x, y, \underline{\psi}(y)) dy.$$

Note that $\underline{\psi}$ continuous implies $T(\underline{\psi})$ continuous and that

$$\begin{aligned} \|T(\underline{\psi}) - \underline{g}\| &= \max_{0 \leq x \leq X} \|T(\underline{\psi})(x) - \underline{g}(x)\| \\ &= \max_{0 \leq x \leq X} \left\| \int_0^x \underline{H}(x, y, \underline{\psi}(y)) dy \right\| \\ &\leq MX \\ &\leq b^*. \end{aligned}$$

Hence $T : S \rightarrow S$, as required by Schauder's fixed point theorem.

To show that T is a contraction mapping, take $\underline{\varphi}$ and $\underline{\psi} \in S$. Then

$$\begin{aligned} \|T(\underline{\varphi}) - T(\underline{\psi})\| &= \max_{0 \leq x \leq X} \left\| \int_0^x \underline{H}(x, y, \underline{\varphi}(y)) dy - \int_0^x \underline{H}(x, y, \underline{\psi}(y)) dy \right\| \\ &\leq \max_{0 \leq x \leq X} \int_0^x \|\underline{H}(x, y, \underline{\varphi}(y)) - \underline{H}(x, y, \underline{\psi}(y))\| dy \\ &\leq \max_{0 \leq x \leq X} L \int_0^x \|\underline{\varphi}(y) - \underline{\psi}(y)\| dy \\ &\leq X \max_{0 \leq x \leq X} L \|\underline{\varphi}(x) - \underline{\psi}(x)\| \\ &= XL \|\underline{\varphi} - \underline{\psi}\|_{\infty} \\ &\leq \alpha \|\underline{\varphi} - \underline{\psi}\|_{\infty}. \end{aligned}$$

Thus, by the contraction mapping principle there is a unique function

$\underline{f} \in S$ with

$$T(\underline{f})(x) = \underline{f}(x) = \underline{g}(x) + \int_0^x \underline{H}(x, y, \underline{f}(y)) dy.$$

1.9.2 Continuation of Solutions

Theorem 1.5 gave us a local existence result for the general non-linear Volterra equation. Here we investigate how large the interval of existence can be made. We require the following definition:

Definition 1.1

Let $g : [0, \infty) \rightarrow (-\infty, \infty)$ and for

$$B = \{(x, y, w) : 0 \leq y \leq x < \infty, w \in \mathbb{R}\}$$

let $H : B \rightarrow (-\infty, \infty)$. Let $f(x)$ be a continuous solution of the scalar equation

$$f(x) = g(x) + \int_0^x H(x, y, f(y)) dy \quad (1.77)$$

on $[0, A]$ with the property that if $u(x)$ is any other solution, then as long as $u(x)$ exists and $x \leq A$ we have $u(x) \leq f(x)$. Then $f(x)$ is called the maximal solution of (1.77).

Of the many useful results which can be proved about maximal solutions, the following (see Burton [13, p.83]) are of particular interest to us.

We observe that if g is continuous on $[0, \infty)$ and $H(x, y, v)$ is continuous for $0 \leq y \leq x < \infty$ and all $v \in \mathbb{R}$ it may be shown that solutions of (1.77) that remain bounded are continuable to all of $[0, \infty)$; cf. Burton [13, p.80].

Theorem 1.6

Let the maximal solution $\hat{f}(x)$ of the scalar equation

$$\hat{f}(x) = c + \int_0^x H(y, \hat{f}(y)) dy$$

exist on $[0, A]$, where c is a constant and let $H : [0, A] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous and non-decreasing in \hat{f} when $0 \leq x \leq A$. If $u(x)$ is a continuous scalar function on $[0, A]$ satisfying

$$u(x) \leq u_0 + \int_0^x H(y, u(y)) dy, \quad u_0 \leq c$$

Then $u(x) \leq \hat{f}(x)$ on $[0, A]$.

The following theorem requires Theorem 1.6 for its proof and it is an extension of the classical result for ordinary differential equations known as the Conti-Wintner theorem.

Theorem 1.7

Let $\underline{g} : [0, \infty) \rightarrow \mathbb{R}^n$ and $\underline{H} : B \rightarrow \mathbb{R}^n$ be continuous where where $B = \{x, y, \underline{w} : 0 \leq y \leq x < \infty; \underline{w} \in \mathbb{R}^n\}$. Suppose that for each $X > 0$ there is a constant $K(X) > 0$ and a continuous function $\omega : [0, \infty) \rightarrow [1, \infty)$ with ω non-decreasing, and

$$\|\underline{H}(x, y, \underline{w})\|_\infty \leq K(X)\omega(\|\underline{w}\|_\infty) \quad \text{if } 0 \leq y \leq x < X \quad \text{and} \quad \int_1^\infty \frac{ds}{\omega(s)} = \infty.$$

If $\underline{f}(x)$ is a solution of

$$\underline{f}(x) = \underline{g}(x) + \int_0^x \underline{H}(x, y, \underline{f}(y)) dy$$

on any interval $[0, \alpha)$ then it is bounded and hence is continuable to $[0, \infty)$.

Proof (The proof may be found in Burton [13, p.83].)

Because $\underline{f}(x)$ is defined on $[0, \alpha)$, take $X = \alpha$. Then

$$\|\underline{f}(x)\|_\infty \leq \|\underline{g}(x)\|_\infty + \int_0^x \|\underline{H}(x, y, \underline{f}(y))\|_\infty dy$$

$$\leq M + \int_0^x K(\alpha) \omega(\|f(y)\|_\infty) dy$$

where $\|g(x)\|_\infty \leq M$ on $[0, \alpha]$.

As ω is monotone, $\|f(x)\|_\infty$ is bounded (Theorem 1.6) by the maximal solution of

$$u(x) = M + \int_0^x K(\alpha) \omega(u(s)) ds,$$

or equivalently of the initial value problem

$$u' = K(\alpha) \omega(u), \quad u(0) = M.$$

On separating the variables this yields

$$\int_M^{u(x)} \left[\frac{ds}{\omega(s)} \right] = K(\alpha)x.$$

As $\int_M^\infty \frac{ds}{\omega(s)} = \infty$, $u(\alpha)$ exists and is finite; hence $f(\alpha)$ is bounded. Since α is arbitrary, $f(x)$ is bounded, hence continuable.

Nohel in [44] has also proved a theorem which is an extension of the comparison technique for ordinary differential equations due to Conti and Wintner, where, for simplicity of presentation, he takes $H(x, y, f) = k(x - y)\varphi(y, f)$. For other applications of Nohel's Theorem we may refer to [43] and in [44] other deeper results on existence and boundedness are stated for some specific problems.

1.9.3 Periodic Solutions

Periodicity of solutions is a subject of interest which we shall not pursue except to give a flavour.

One such problem arising from the theory of super fluidity and developed by C.C. Lin is also discussed by Nohel [44]. The problem has also been studied by Levinson [34] and illustrates some properties of periodic solutions. Here, the heat equation, because of a

complicated boundary initial value problem, leads to the integral equation

$$f(x) + g(x) = \frac{-1}{\sqrt{\pi}} \int_0^x \frac{\Phi(f(\tau))}{\sqrt{(x-\tau)}} d\tau \quad x \geq 0 \quad (1.78)$$

where g is a given periodic function.

The following results are due to Levinson [34].

Theorem 1.8

Let $g(x)$ be continuous on $0 \leq x < \infty$ and satisfy a uniform Hölder condition of exponent $\beta > 0$ on any finite interval. Let $\Phi(y)$ be monotone increasing on $-\infty < y < \infty$, $\Phi(0) = 0$; let Φ satisfy a local Lipschitz condition. Then (1.78) possesses a unique solution $f(x)$ on $(0, \infty)$.

Theorem 1.9

Let the hypothesis of Theorem 1.8 be satisfied and in addition let $g(x)$ have period ω and let $\max |g(x)| = M$. Suppose there exists a positive monotone increasing function $k(u)$ for $u > 0$ such that

$$\Phi(y_2) - \Phi(y_1) \geq k(y_2 - y_1)$$

for $y_2 - y_1 > 0$ and $|y_1|, |y_2| \leq 2M$. Then there exists a continuous periodic function $\varphi(x)$ of period ω such that the solution $f(x)$ of (1.78) satisfies

$$\lim_{x \rightarrow \infty} (f(x) + g(x) - \varphi(x)) = 0.$$

Moreover $|f(x) + g(x)| \leq \max |g(x)|$.

This result establishes asymptotic periodicity.

1.9.4 Stability

Just as for differential equations we investigate different types

of stability for Volterra integral equations. Essentially, we study the sensitivity of $f(x)$ to perturbations in the problem (say in $g(x)$) in particular as $x \rightarrow \infty$. For an introduction to this topic we refer to Tsalyuk [51]. However, there is a proliferation of terminology in the literature and the word 'stability' is used with varying interpretations. Thus, we state here a series of fundamental definitions. Furthermore, we confine our attention in this and subsequent sections in the introduction to *inherent* stability (as opposed to stability of a numerical scheme). Following the literature we consider (abstract) Volterra equations of the second kind:

$$f(x) - \int_a^x K(x, y, f(y)) dy = g(x), \quad x \geq 0 \quad (1.79)$$

where $f, g : [0, \infty) \rightarrow X$, $((X, \|\cdot\|))$ may be a Banach Space, but we are concerned with the choices $X = \mathbb{R}$ or $X = \mathbb{R}^n$ and

$$K : [0, \infty)^2 \times \{f \in X : \|f\| < r\} \rightarrow X \quad \text{and} \quad K(x, y, f) = 0 \quad \text{for} \quad y > x.$$

We shall assume $K(x, y, 0) \equiv 0$, and frequently take $g(x) \equiv 0$, to consider small non-zero perturbations in this g . In the definitions which follow let M_0 and N_0 be some subsets of the normed spaces of mappings of $[0, \infty)$ into X . (Most frequently, in the literature,

$$M_0 = BC([0, \infty) \rightarrow \mathbb{R}^n) = \{f \equiv \underline{f} \in C([0, \infty) \rightarrow \mathbb{R}^n) : \|f\| = \sup_{0 \leq x < \infty} \|\underline{f}(x)\| < \infty\}$$

and N_0 is a linear sub-space of $C([0, \infty) \rightarrow \mathbb{R}^n)$.)

Definition 1.2

The trivial solution (for $g \equiv 0$) of (1.79) is said to be stable for a given $a \in [0, \infty)$ if

(i) there exists $\delta_0 > 0$ such that for any $g \in N_0$, $\|g\| < \delta_0$ (1.79) has a solution $f \in M_0$;

(ii) for any $\epsilon > 0$ there exists $\delta(\epsilon, a) > 0$ such that $g \in N_0$, $\|g\| < \delta \Rightarrow \|f\| < \epsilon$.

The trivial solution of (1.79) is said to be uniformly stable relative to the parameter in (N_0, M_0) if in condition (ii) δ can be chosen independently of the parameter. In particular, the lower limit a is such a parameter.

Definition 1.3

Assume that $P_0 \subset N_0$ is a subset of some normed space. The trivial solution of (1.79) is said to be asymptotically stable in (N_0, M_0, P_0) if it is stable in (N_0, M_0) and there exists $\delta_1 > 0$ such that $g \in P_0$, $\|g\|_{P_0} < \delta_1 \Rightarrow \lim_{x \rightarrow \infty} f(x) = 0$.

Depending on the choice of N_0 and P_0 , various forms of uniform stability are possible. Assume, for example, that

$$P_0 = C_0([0, \infty) \rightarrow \mathbb{R}^n) = \{f \in BC : \lim_{x \rightarrow \infty} f(x) = 0\}.$$

Definition 1.4

The trivial solution of (1.79) is uniformly asymptotically stable in (BC, BC, C_0) if it is stable in (BC, BC) and if for some $\delta_0 > 0$ for any $\epsilon > 0$ and $T \geq a$ there exists $\delta(\epsilon, T)$ and $T_1(\epsilon, T)$ such that for any $g \in C_0$, satisfying $\|g\|_{C_0} < \delta_0$, $\|g(x)\| < \delta$ for $x \geq T$, the solution of (1.79) satisfies the inequality $\|f(x)\| < \epsilon$ for $x \geq T_1$.

For references to other types of stability which have been studied see Tsalyuk [51]. We give here one further definition, that of strong stability, in the sense of Bownds and Cushing [8].

Definition 1.5

The trivial solution of (1.79) is called strongly stable on $P_0 \subset C([0, \infty) \rightarrow R^n)$ if for any $\epsilon > 0$ and $a \in [0, \infty)$ there exists $\delta(\epsilon) > 0$ such that from $g \in P_0$ and $\|f(x_1)\| \leq \delta$ for some $x_1 \geq 0$ there follows $\|f(x)\| \leq \epsilon$ for all $x > 0$.

1.9.5 Stability criteria in terms of the resolvent or characteristics of the kernel

We here consider the linear equation

$$f(x) = \int_0^x K(x, y)f(y)dy + g(x) \quad (1.80)$$

which we have already observed in (1.71) may be represented in operator theory as

$$(I - K)f = g.$$

It follows that (i) (1.80) is stable in (N_0, M_0) if and only if the operator $(I - K)^{-1}$ acts from N_0 into M_0 and is continuous and (ii) (1.80) is asymptotically stable in (N_0, M_0, P_0) if $(I - K)^{-1}$ acts continuously from N_0 into M_0 and also $(I - K)^{-1}P_0$ is contained in the set of functions which tend to zero for $x \rightarrow \infty$.

We examine first the stability criteria which may be expressed in terms of the resolvent. For convenience, we recall that for $g \in C([0, \infty) \rightarrow R^n)$ the solution for $f \in C([0, \infty) \rightarrow R^n)$ and that the kernel K admits a resolvent $R(x, y)$ and a differential resolvent $U(x, y)$ (see §1.6.1) such that for the solutions of (1.80) we have

$$f(x) = g(x) + \int_0^x R(x, u)g(u)du \quad (1.81)$$

and

$$f(x) = U(x, 0)g(0) + \int_0^x U(x, y)g'(y)dy \quad (1.82)$$

where $U(x, y) = 1 + \int_y^x R(x, z)dz.$

To illustrate the sensitivity of $f(x)$ to perturbations let us suppose, for example, that in (1.80) $K(x, y) \geq M > 0$ for $0 \leq y \leq x$ and that $g(x)$ is perturbed to a function $g(x) + \delta g(x)$ and that the corresponding solution changes from $f(x)$ to $f(x) + \delta f(x)$ where

$$\delta f(x) = \int_0^x K(x, y) \delta f(y) dy + \delta g(x). \quad (1.83)$$

Then

$$\delta f(x) = \delta g(x) + \int_0^x R(x, u) \delta g(u) du \quad (1.84)$$

where $R(x, y)$ is the resolvent given in (1.81) and which is also developed from the Neumann Series $R(x, y) = \sum_{j=1}^{\infty} K^j(x, y)$, $K^1(x, y) = K(x, y)$ where $K^j(x, y) = \int_0^x K(x, z) K^{j-1}(z, y) dz$. It may be shown, by induction, that

$$K^p(x, y) \geq \frac{M^p (x - y)^{p-1}}{(p - 1)!}$$

and hence that $R(x, y) \geq M e^{M(x-y)}$.

Hence, from (1.84)

$$\delta f(x) \geq \delta g(x) + M \int_0^x e^{M(x-y)} \delta g(y) dy$$

when $\delta g(x) > 0$. Thus $\delta f(x)$ has a possibly increasing component behaving like $M e^{Mx} \int_0^x e^{-My} \delta g(y) dy$ and (1.80) is susceptible to ill-conditioning.

One may express stability criteria in terms of the resolvent or in terms of criteria related to the characteristics of the kernel itself. Here, we list briefly two statements on stability which are true in terms of the resolvent; for others we refer to Tsalyuk [51].

1. Equation (1.80) is stable in (BC, BC) if and only if

$$\sup_{x \geq 0} \int_0^x \|R(x, y)\| dy < \infty. \quad (1.85)$$

2. Equation (1.80) is stable (or uniformly stable relative to a) in (\mathbb{R}^n, BC) if and only if

$$\sup_{x \geq 0} \|U(x, 0)\| < \infty,$$

$$(\text{or } \sup_{x, y \geq 0} \|U(x, y)\| < \infty). \quad (1.86)$$

We deduce that if the equation

$$\tilde{f}(x) = \int_0^x \tilde{K}(x, y) \tilde{f}(y) dy + \tilde{g}(x), \quad (1.87)$$

when $\tilde{K}(x, y)$ is a matrix with elements $\tilde{K}_{ij}(x, y)$, is stable in (BC, BC) then so also is (1.80).

For kernels $K(x, y) \geq 0$ the following statements concerning stability may be made. These are examples of many others which may be found by reference to Tsalyuk [51].

1. For the stability of (1.80) in (BC, BC) it is necessary and sufficient that

$$\sup_{x \geq 0} \int_0^x \|K(x, y)\| dy < \infty$$

and that for some ν the spectrum of the matrix

$$A_\nu = \lim_{X \rightarrow \infty} \limsup_{x \rightarrow \infty} \int_x^X K^\nu(x, y) dy = \lim_{X \rightarrow \infty} \sup_{x \geq X} \int_x^X K^\nu(x, y) dy$$

where $K^\nu(x, y)$ is the ν -th iterated kernel, should lie in the unit circle.

2. If equation (1.80) is stable in (BC, BC) then for any ν the spectrum of the matrix

$$B_\nu = \lim_{T \rightarrow \infty} \liminf_{t \rightarrow \infty} \int_T^t K^\nu(x, y) dy$$

lies in the unit circle.

1.9.6 Convolution kernels revisited

Of special interest are the equations in which the kernel is

convolution where the resolvent $R(x, y) = r(x - y)$ and the stability of

$$f(x) - \int_0^x k(x - y)\varphi(f(y))dy = g(x) \quad (1.88)$$

in (BC, BC) is equivalent to the requirement

$$\int_0^\infty \|r(y)\| dy < \infty. \quad (1.89)$$

Equation (1.88) has been studied under various assumptions on $k(x)$, $g(x)$ and $\varphi(x)$. The particular case where $k(x)$ and $g(x)$ are non-negative functions and $\varphi(x) \equiv x$ gives rise to the renewal equation which has attracted interest in many areas. We refer to the account of Bellman and Cooke [7], for example, for a review of 41 problems of historical interest related to the renewal equation. The existence of a solution $f(x)$ of

$$f(x) - \int_0^x k(x - y)f(y)dy = g(x) \quad (1.90)$$

and of its stability can be deduced from a well known result of Paley and Wiener and its association with transform theory.

Theorem 1.10 (Paley and Wiener [45])

If, in equation (1.90), $\int_0^\infty |k(t)|dt < \infty$, then a necessary and sufficient condition in order that (1.89) is satisfied is that $L(k)(z) \neq 1$ for $\operatorname{Re} z \geq 0$ where $L(k)(z) = \int_0^\infty k(t) e^{-zt} dt$.

The relationship with stability follows from (1.31c):

$$f(x) = g(x) + \int_0^x r(x - u)g(u)du$$

and the observation that

$$\sup_{x \geq 0} |f(x)| \leq \sup_{x \geq 0} |g(x)| \left\{ 1 + \int_0^\infty |r(u)| du \right\}.$$

Thus $\sup|f(x)|$ is finite when $\sup|g(x)|$ is finite *given that all the conditions of the Paley-Wiener theorem are satisfied.*

The classical paper by Feller [25] in 1941 attempted to resolve some of the controversies and conjectures at that time and to correct some of the previously announced results. Feller did not use the result of Paley and Wiener as he argued that this result was too deep for the properties of $k(x)$ and $g(x)$ which concerned him. Feller's main objective was not to study the behaviour of $f(x)$ as $x \rightarrow \infty$ but to study the asymptotic behaviour of the mean value $f^*(x) = \frac{1}{x} \int_0^x f(x)dx$. This was because, whilst it was generally supposed that $f(x)$ behaved like an exponential function or that it approached in an oscillating manner a finite limit q (the latter case should arise if $\int_0^\infty k(y)dy = 1$), it was possible to construct specific examples in which $f(x)$ did not behave in this manner. Feller dealt with two particular applications of (1.88). The first in the theory of industrial replacement, formulated by Lotka, where it is assumed that each individual dropping out is immediately replaced by a new member of zero age. Here $\int_0^\infty k(y)dy = 1$ where $k(y)$ represents a density of probability. The second formulation is one in which $f(x)$ measures the rate of female birth at time $x > 0$. That is, if $k(x)$ represents the reproduction rate at age x then the average number of females born during $(x, x+\delta x)$ from a female of age x is $k(x)\delta x + O(\delta x)$. If $\eta(y)$ stands for the age distribution of the parent population at $x = 0$ then

$$g(x) = \int_0^x \eta(y)k(x-y)dy$$

measures the rate of production of females at time x by members of the parent population. Then if $f(x)$ measures the rate of female birth at $x > 0$ we arrive at the following:

* The parent population comprises those mothers born before $x=0$, that is, $x-y < 0$

$$f(x) = g(x) + \int_0^x f(x-y)k(y)dy \quad (1.91) \quad *$$

This time k is not a probability density function and $\int_0^\infty k(y)dy \begin{matrix} > \\ < \end{matrix} 1$ is a non-negative number measuring the tendency of the population to increase or decrease.

We state without proof two of the results of Feller on asymptotic properties of (1.91).

Theorem 1.11

Assume $k(x) > 0$, $g(x) > 0$ and suppose that $\int_0^\infty k(y)dy = a$, $\int_0^\infty g(y)dy = b$ where a and b are finite.

- (i) In order that $f^*(x) = \frac{1}{x} \int_0^x f(y)dy \rightarrow C$ as $x \rightarrow \infty$, where C is a positive constant, it is necessary and sufficient that $a = 1$ and that $\int_0^\infty yk(y)dy = m$, a finite number. In this case $C = b/m$.
- (ii) If $a < 1$, then $\int_0^\infty f(x)dx = \frac{b}{1-a}$.

Next we deal with the important special case when $\int_0^\infty k(y)dy = 1$.

Theorem 1.12

Let $\int_0^\infty k(y)dy = 1$, $\int_0^\infty g(y)dy = b < \infty$. Suppose there is an integer $n \geq 2$ such that $m_k = \int_0^\infty x^k k(x)dx$, ($k = 1, 2, \dots, n$), are finite and that the functions $k(x)$, $xk(x)$, \dots , $x^{n-2}k(x)$ are of bounded total variation over $(0, \infty)$. Suppose also that $\lim_{x \rightarrow \infty} x^{n-2}g(x) = 0$ and $\lim_{x \rightarrow \infty} x^{n-2} \int_x^\infty g(s)ds = 0$.

Then $\lim_{x \rightarrow \infty} f(x) = \frac{b}{m_1}$ and $\lim_{x \rightarrow \infty} x^{n-2} [f(x) - \frac{b}{m_1}] = 0$.

1.9.7 Asymptotic behaviour of solutions of integro-differential equations

In his review, because of the length restriction on his survey, Tsalyuk [51] does not refer to papers on Volterra integro-differential equations. The equation

$$f'(x) = \int_0^x a(x-y)\phi(f(y))dy \quad (0 \leq x < \infty) \quad (1.92)$$

* The integral on the right hand side of (1.91) refers to births from mothers born after $x=0$, that is, $x-y > 0$.

may, of course, be converted into the Volterra integral equation (1.88) by integration in which $k(x-y) = \int_y^x a(u-y) du$ and $g(x) \equiv f(0)$. However, integro-differential equations of the form (1.92) are frequently best studied directly rather than as conversions to the form (1.88). The equation (1.92) arises in nuclear reactor dynamics under hypotheses:

- (i) $a(x) \in C[0, \infty)$, $(-1)^s a^{(s)}(x) \geq 0$ ($0 < x < \infty$; $s = 0, 1, 2, 3$),
- (ii) $\varphi(x) \in C(-\infty, \infty)$, $x\varphi(x) > 0$ ($x \neq 0$)

and has been studied extensively, particularly by Levin and Nohel.

In earlier work they studied the asymptotic behaviour of (1.92) in the case when $\varphi(x) \equiv x$. Their situation was different from that considered by Feller as the functions corresponding to $k(x)$ and $g(x)$ in (1.90) were not integrable over $(0, \infty)$ nor were they non-negative. In fact, to investigate asymptotic properties, Levin and Nohel used a Tauberian theorem for Laplace transforms. However, such methods do not lend themselves to non-linear problems. Levin and Nohel have also studied (1.92) in the non-linear case; in the reactor problem $\varphi(x) = e^x - 1$. Levin [31] has established the following result.

Theorem 1.13

Let (i) and (ii) be satisfied. Then given any f_0 there exists a solution $f = u(x)$, $u(0) = f_0$ on $0 \leq x < \infty$. Moreover there exists a constant $K = K(f_0)$ such that

$$|u^{(j)}(x)| \leq K \quad (j = 0, 1, 2; \quad 0 \leq x < \infty).$$

If also $a(x) \neq a_0$ then

$$\lim_{x \rightarrow \infty} u^{(j)}(x) = 0 \quad (j = 0, 1, 2).$$

Nohel [44] comments that, in the case $\varphi(x) = e^x - 1$, $\varphi(x)$ satisfies (ii), $a(x)$ not only satisfies (i) but also satisfies

$$(-1)^s a^{(s)}(x) \geq 0 \quad (0 < x < \infty, s = 0, 1, \dots)$$

i.e. $a(x)$ is completely monotonic on $[0, \infty)$.

We note that when the linearity assumption $\varphi(x) \equiv x$ is dropped then Levin [32] has obtained a non-linear generalization of the Paley-Wiener Theorem. Another non-linear generalization of the same theorem due to Miller [40] will be referred to in the following section.

Many other problems which lead to integro-differential equations may be found in Nohel [44] where an extensive bibliography is provided.

1.9.8 The Linearization of Volterra Integral Equations

A further gap in the theory of Volterra integral equations has been investigated by Miller [40] who has developed a theory of linearization of Volterra integral equations which is an analogue of a theory of asymptotic stability of differential equations. For the non-linear differential equation

$$y' = Cy + O(|y|) \quad (1.93)$$

it is well known that the asymptotic stability of the linear system $x' = Cx$ implies the local asymptotic stability of the trivial solution of (1.93). Now, we consider the system of equations

$$f(x) = g(x) + \int_0^x a(x-y)\varphi(f(y))dy \quad x \geq 0 \quad (1.94)$$

where f, g, φ are n -vectors, $a(x)$ is an $n \times n$ matrix and $\varphi(0) = 0$. If g is "small" this system is often replaced by the more easily analysed linear system

$$y(x) = g(x) + \int_0^x a(x-s)Jy(s)ds \quad (1.95)$$

where J is the Jacobian matrix $\varphi'(0) = (\partial\varphi_i(0)/\partial f_j)$.

Nohel [44] drew attention to a gap in the theory of Volterra integral equations indicating that, except in the case when solutions of (1.95) decay exponentially, it had not been possible to show that solutions of the linear system (1.95) approximate those of (1.94).

(Nohel observes that certain problems in reactor dynamics give rise to the situation where solutions need not decay exponentially.)

Miller [40] has produced such a theory of linearization of (1.94) under very general assumptions on $a(x)$, J . ^{Using Theorem 1 of [39],} the method produced consists of replacing the local non-linear problem (1.94) by the linear equation (1.95) and using the linear equation for its resolvent. These latter equations may be studied, for example, by transform techniques.

For (1.94), the following assumptions are made:

- (i) $a \in L^1(0, \infty)$ for each $x > 0$; (ii) $g(x) \in C[0, \infty)$;
- (iii) $\varphi(x) \in C^1(\mathbb{R}^n)$, $\varphi(0) = 0$; (iv) the Jacobian matrix J is non-singular; (v) $|r(x)| \in L^1(0, \infty)$ for all $x > 0$, where $r(x)$ is the resolvent kernel of

$$y(x) = g(x) + \int_0^x a(x-s)y(s)ds \quad (1.95')$$

Miller has derived the following general stability results:

Theorem 1.14

Given that assumptions (i)-(v) are satisfied there exists $\varepsilon_0 > 0$ and $\varepsilon_1 > 0$ such that when the solution $y(x)$ of (1.95') satisfies $\|y\|_0 < \varepsilon_0$ the solution $f(x)$ of (1.94) exists for all $x \geq 0$ and $\|f\|_0 < \varepsilon_1$. ($h(t) \in BC[0, \infty)$; $\|h\|_0 = \sup \{|h(t)|; 0 \leq t < \infty\}$).

Theorem 1.15

Given that assumptions (i)-(v) hold and $\varepsilon_0, \varepsilon_1$ are given by Theorem 1.14 above then if $\|y\|_0 < \varepsilon_0$ and $y(x) \rightarrow 0$ as $x \rightarrow \infty$ then $f(x) \rightarrow 0$ as $x \rightarrow \infty$.

With the additional assumption that $a \in L^1(0, \infty)$, Miller [40] has produced the following non-linear generalization of the Paley-Wiener Theorem to which we have previously referred.

Theorem 1.16

Suppose that (i)-(iv) hold; $\det(I - \int_0^\infty \exp(-st)a(t)dt) \neq 0$ for $\operatorname{Re} s \geq 0$ and $\epsilon_2 : \epsilon_2(1 + \int_0^\infty |r(s)|ds) < \frac{\epsilon_1}{2}$ where ϵ_1 is the constant given in Theorem 1.14. If $\|g\|_0 < \epsilon_2$ and $g(t) \rightarrow 0$ as $t \rightarrow 0$ then $f(x) \rightarrow 0$.

In subsequent sections in [40], Miller studies applications of the theory to integro-differential equations and, in particular, to the reactor problem. However, the final theorem of [40] is neither stronger nor weaker than the results of Levin and Nohel in [33] although it produces a result which is local while the results of [33] are global.

REFERENCES

1. AMINI, S., BAKER C.T.H., van der HOUWEN, R.J. & WOLKENFELT, P.H.M. Stability analysis of numerical methods for Volterra integral equations with polynomial convolution kernels. Rep. NW 109/81 Math. Centrum, Amsterdam (1981).
2. ANDERSSEN, R.S. & de HOOG, F.R. Application and numerical solution of Abel-type integral equations. Mathematics, Faculty of Science, ANU Technical Report (Proceedings of A Delaware Conference 1980).
3. ANSELONE, P.M. (ed) Non linear Integral Equations. Univ. of Wisconsin Press (1964).
4. BAKER, C.T.H. The numerical treatment of integral equations. Clarendon Press, Oxford (1977, reprinted 1978).
5. BAKER, C.T.H. & PHILLIPS, C. (eds) The numerical solution of non-linear problems. Clarendon Press, Oxford (1981).
6. BARUCHA-REID, A.T. Random Integral Equations. Vol. 96 in Mathematics in Science and Engineering. Academic, (1972).
7. BELLMAN, R. & COOKE, K.L. Differential-Difference equations. Academic, New York (1963).
8. BOWND, J.M. & CUSHING, J.M. Some stability theorems for systems of Volterra integral equations. Appl. Anal. 5 pp. 65-77 (1975).
9. BRAUER, F. A nonlinear variation of constants formula for Volterra equations. Math, Systems Theory, Vol 6, No.3 pp.226-234 (1972).
10. BRAUER, F. On a nonlinear integral equation for population growth problems. Siam J. Math. Anal. Vol 6, No. 2, pp. 312-317 (1975).
11. BRUNNER, H. The application of the variation of constants formulas in the numerical analysis of integral and integro-differential equations. Utilitas Mathematica, Vol 19, pp. 255-290 (1981).
12. BRUNNER, H and van der Houwen, P.J. The Numerical solution of Volterra equations. CWI Monograph 3, North Holland (1986).
13. BURTON, T.A. Volterra integral and differential equations. Vol 167 in Mathematics and Science and Engineering. Academic (1983).
14. BURTON, T.A. Structure of Solutions of Volterra Equations. Siam Review Vol 25, No. 3 pp.343-364 (1983).
15. CHENEY, E.W. Introduction to Approximation Theory. McGraw-Hill, New York (1966).
16. COCHRAN, J.A. Analysis of linear integral equations. McGraw-Hill, New York (1979).

17. COLLATZ, L. Functional Analysis and Numerical Mathematics. Academic Press, London (1966).
18. CORDUNEANU, C. Principles of differential and integral equations. Allyn & Bacon, Boston, Massachusetts (1971).
19. COURANT, R. & HILBERT, D. Methods of mathematical physics. Vol 1, Interscience, New York (1953).
20. CUSHING, J.M. Forced asymptotically periodic solutions of predator-prey systems with or without hereditary effects. Siam J. Appl. Math. Vol 30. pp 665-674 (1976).
21. DAVIS, H.T. Introduction to nonlinear differential and integral equations. First published United States Atomic Energy Commission (1960); also Dover, New York (1962).
22. DIEKMANN, O. Thresholds and travelling waves for the geographical spread of infection. J. Math. Biology, Vol 6, 109-130 (1978).
23. DOETSCH, G. Introduction to the theory and application of the Laplace transformation. Springer, New York (1978).
24. FELLER, W. Completely monotone functions and sequences. Duke Math. Jour., Vol 5, pp. 661-674 (1939).
25. FELLER, W. On the integral equation of renewal theory. Annals of Math. Stat., Vol 12, pp. 243-267 (1941).
26. GOLBERG, M.A. (ed) Solution methods for Integral Equations. Plenum Press, New York (1979).
27. GOLBERG, M.A. On a method of Bounds for solving Volterra integral equations. pp. 245-246 in [26]. GOLBERG, M.A. (ed).
28. HETHCOTE, H.W. & van den DRIESSCHE, P. Stability Analysis for models of diseases without immunity. J. Math. Biology, Vol 13, pp. 185-198 (1981).
29. HETHCOTE, H.W. & TUDOR, D.W. Integral Equation models for endemic infectious diseases. J. Math. Biology, Vol 9, pp. 37-47 (1980).
30. KERSHAW, D. Functional equations, fixed points and the Newton-Kantorovitch Theory pp.32-42 in [5]. BAKER, C.T.H. & PHILLIPS, C. (eds).
31. LEVIN, J.J. The asymptotic behaviour of the solution of a Volterra equation. Proc. Amer. Math. Soc., Vol 14, pp. 534-541 (1963).
32. LEVIN, J.J. The qualitative behaviour of a nonlinear Volterra equation. Proc. Amer. Math. Soc., Vol 16, pp. 711-718 (1965).
33. LEVIN, J.J. & NOHEL, J.A. A system of non linear integro-differential equations. Mich. Math. J. Vol 13, pp. 257-270 (1966).

34. LEVINSON, N. A non linear Volterra equation arising in the theory of superfluidity. J. Math. Anal. and Appl., Vol 1, pp. 1-11 (1961).
35. LONSETH, A.T. Sources and applications of integral equations. Siam Review, Vol 19, pp. 241-278 (1977).
36. LOTKA, A.J. A contribution to the theory of self-renewing aggregates, with special reference to industrial replacement. Annals of Math. Stat., Vol 10, pp. 1-25 (1939).
37. LOTKA, A.J. On an integral equation in population analysis. Annals of Math. Stat., Vol 10, pp. 144-161 (1939).
38. MIKHLIN, S.G. Integral equations and their applications to some problems of mechanics, mathematical physics and engineering. Pergamon Press, Oxford (1964).
39. MILLER, R.K. Asymptotic behaviour of solutions of non linear Volterra equations. Bull. Amer. Math. Soc., Vol 72, pp. 153-156 (1966).
40. MILLER, R.K. On the linearization of Volterra integral equations. J. Math. Anal. and Appl., Vol 23, pp. 198-208 (1968).
41. MILLER, R.K. Non linear Volterra integral equations. Benjamin, Menlo Park (1971).
42. NOBLE, B. Instability when solving Volterra integral equations of the second kind by multistep methods. Springer Lect. Notes Math., Vol 109, pp. 23-39 (1969).
43. NOHEL, J.A. Some problems in non linear Volterra equations. Bull. Amer. Math. Soc., Vol 68, pp. 323-329 (1962).
44. NOHEL, J.A. Problems in qualitative behaviour of solutions of non linear Volterra equations. pp. 191-214 in [3]. ANSELONE, P.M. (ed).
45. PALEY, R.E.A.C. & WIENER, N. Notes on the theory and application of Fourier transforms, VII On the Volterra equation. Amer. Math. Soc. Trans., Vol 35, pp. 785-791 (1933).
46. POUZET, P. Étude en vue de leur traitement numérique des équations intégrales de type Volterra. Revue Fr. Trait. Inf., 6 pp. 79-112 (1963).
47. SAATY, T.L. Modern nonlinear equations. McGraw-Hill, New York (1967).
48. SMITHIES, F. Integral Equations. Cambridge University Press, Cambridge (1962).
49. THIELMAN, H.P. Applications of fixed point theorems by Russian mathematicians. pp. 35-48 in [3]. ANSELONE, P.M. (ed).
50. TRICOMI, F.G. Integral equations. Interscience, New York (1957).

51. TSALYUK, Z.B. Volterra integral equations. J. Soviet Math., Vol 12, pp. 715-758 (1979).
52. WATSON, E.J. Laplace transforms and applications. van Nostrand, New York (1981).
53. WOLKENFELT, P.H.M. Modified multilag methods for Volterra functional equations. Rep. NW 108/81, Math. Centrum, Amsterdam (1981).
54. WOLKENFELT, P.H.M. The numerical analysis of reducible quadrature methods for Volterra integral and integrodifferential equations. Acad. Proefschrift, Amsterdam (1981).
55. WOLKENFELT, P.H.M. van der HOUWEN, P.J. & BAKER, C.T.H. Analysis of numerical methods for second kind Volterra equations by imbedding techniques. J. Int. Eqns. Vol 3, pp. 61-82 (1981).
56. WOUK, A. Direct Iteration, Existence and Uniqueness. pp. 3-33 in [3]. ANSELONE, P.M. (ed).
57. ZEMANIAN, A.H. Distribution Theory and Transform Analysis. McGraw-Hill, New York (1965).

CHAPTER 2

2. Methods for non-singular problems

- 2.1 Preliminary Remarks
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CHAPTER 2

2.1. Preliminary Remarks

In this chapter we study methods for the numerical solution of linear and non-linear Volterra equations of the second kind and linear Volterra equations of the first kind. The first half of the chapter is largely devoted to a description of numerical methods, with examples; whilst the remainder of the chapter consists of a theoretical investigation of such methods. The techniques for Volterra equations may be divided, roughly, into quadrature methods, classical Runge-Kutta methods and modifications of these. The ("Pouzet")-Runge-Kutta methods themselves may be varied as an extension of quadrature methods, as one-step methods which are modifications of their counterparts in ordinary differential equations and as extensions of piecewise-polynomial collocation methods.

2.2 Quadrature Methods

First, we consider numerical methods for Volterra equations of the second kind:

$$f(x) - \int_0^x H(x,y,f(y))dy = g(x), \quad x > 0 \quad (2.1)$$

where g and H are prescribed and satisfy convenient smoothness assumptions; we seek f . Associated with the methods is a discretization parameter h .

All methods considered are convergent as $h \rightarrow 0$ under a general set of conditions on $H(x,y,v)$ and $g(x)$ but it is known that not all methods are suitable in practice.

Various methods for the numerical solution of (2.1) upon a "mesh" $\{\tilde{\tau}_i\}$ may be derived by setting $x = \tilde{\tau}_i$ in (2.1) and discretizing. Thus, if for convenience, we choose a fixed step length $h > 0$ and set $x = \tilde{\tau}_i = ih$ in (2.1) we obtain

$$f(ih) - \int_0^{ih} H(ih, y, f(y)) dy = g(ih) \quad (2.2)$$

We define a set of quadrature rules

$$Q : \int_0^{ih} \varphi(ih, y) dy \approx h \sum_{k=0}^i \omega_{ik} \varphi(ih, kh) \quad (h > 0, i=1, 2, 3, \dots) \quad (2.3)$$

We set $\omega_{ik} = 0, k > i$. On applying the rules Q to equation (2.2), we set $\varphi(ih, y) = H(ih, y, f(y))$. The resulting equations yield a quadrature method for (2.1).

Definition 2.1

A quadrature method for (2.1) defined by the rules (2.3) consists of determining values $\tilde{f}_i, i = 1, 2, 3, \dots$ such that

$$\tilde{f}_i - h \sum_{k=0}^i \omega_{ik} H(ih, kh, \tilde{f}_k) = g(ih), \quad i = 1, 2, 3, \dots \quad (2.4)$$

where $\tilde{f}_0 = f(0) = g(0)$.

The equations (2.4) may be solved for $i = 1, 2, 3, \dots$ in turn to yield the approximate values $\tilde{f}_i = \tilde{f}(ih), i = 1, 2, 3, \dots$. For each i there is a non-linear equation in \tilde{f}_i to be solved by an iterative technique. (The equations are linear if H is linear in f or if $\omega_{ii} = 0$.) We may take, as the iterative technique, the method

$$\tilde{f}^{(\nu)}(ih) - h \omega_{ii} H(ih, ih, \tilde{f}^{(\nu-1)}(ih)) = \hat{g}_i \quad (2.5)$$

where

$$\hat{g}_i = g(ih) + h \sum_{k=0}^{i-1} \omega_{ik} H(ih, kh, \tilde{f}(kh)) \quad \text{and}$$

$\tilde{f}^{(0)}(ih)$ is predicted by the use of a formula similar to (2.4) in which the weights $\omega_{ij}^{(0)}$ (say) are chosen with $\omega_{ii}^{(0)} = 0$. It is computationally efficient in the computation of terms like \hat{g}_i if $\omega_{ij}^{(0)} = \omega_{ij}$ for $j = 0, 1, \dots, k(i)$, where $k(i)$ is close to i . Iteration (2.5) converges for h sufficiently small provided $H(x, x, v)$ satisfies a Lipschitz

condition in v , uniformly for $0 \leq x \leq X$. The method (2.4) is called a (step-by-step) direct quadrature method. A rigorous discussion of such methods including their properties and convergence behaviour may be found in Baker [2].

Rules Q may be chosen in a number of ways:

Example 2.1 (Nevanlinna [28]).

"Nevanlinna" rules are described as follows:

$$\begin{array}{ccccccccc} k = 0, & 1, & 2, & 3, & & & i-1, & i \\ \omega_{ik} & 1-\theta, & 1, & 1, & 1, & & 1, & \theta \end{array}$$

The array of weights has the appearance

$1-\theta$	θ						
$1-\theta$	1	θ					
$1-\theta$	1	1	θ				
$1-\theta$	1	1	1	θ			
$1-\theta$	1	1	1	1	θ		

where k runs across the columns from $k = 0$ and i runs down the rows from $i = 1$.

The repeated trapezium rule is given by $\theta = \frac{1}{2}$; the (explicit) Euler rule and the (implicit) backward Euler rule correspond to $\theta = 0$ and $\theta = 1$ respectively. Partitioning lines have a significance discussed later.

Example 2.2 ("Gregory" rules)

Details of these may be found in Baker [2]. We illustrate with two examples.

$$\begin{array}{cccccccccccc} k=0 & 1 & 2 & 3 & 4 & \dots & i-4 & i-3 & i-2 & i-1 & i \\ \text{(a)} \quad \omega_{ik} & \frac{5}{12} & \frac{13}{12} & 1 & 1 & 1 & \dots & 1 & 1 & 1 & \frac{13}{12} & \frac{5}{12} \end{array}$$

($i > 2$) with a special case if $i = 2$.

$$(b) \quad \omega_{ik} \quad \frac{3}{8} \quad \frac{7}{6} \quad \frac{23}{24} \quad 1 \quad 1 \dots\dots 1 \quad 1 \quad \frac{23}{24} \quad \frac{7}{6} \quad \frac{3}{8}$$

($i > 4$) with a special case if $i = 3$.

In practice, the Gregory rules cannot be used until i is sufficiently large and, thus, there is a need for a 'starting procedure'.

(a) Consider the Gregory scheme of order 3, supplemented by

$\omega_{10} = \omega_{11} = \frac{1}{2}$, then the array of weights has the appearance:

0					
$\frac{1}{2}$	$\frac{1}{2}$				
$\frac{5}{12}$	$\frac{14}{12}$	$\frac{5}{12}$			
$\frac{5}{12}$	$\frac{13}{12}$	$\frac{13}{12}$	$\frac{5}{12}$		
$\frac{5}{12}$	$\frac{13}{12}$	1	$\frac{13}{12}$	$\frac{5}{12}$	
$\frac{5}{12}$	$\frac{13}{12}$	1	1	$\frac{13}{12}$	$\frac{5}{12}$
$\frac{5}{12}$	$\frac{13}{12}$	1	1	1	$\frac{13}{12}$
$\frac{5}{12}$	$\frac{13}{12}$	1	1	1	$\frac{13}{12}$
$\frac{5}{12}$	$\frac{13}{12}$	1	1	1	$\frac{13}{12}$

(b) With the second scheme, supplemented by $\omega_{10} = \omega_{11} = \frac{1}{2}$ and

$\omega_{20} = \omega_{22} = \frac{1}{3}$, $\omega_{21} = \frac{4}{3}$, the array of weights has the appearance

D

D

D

D

D

D

D

D

D

D

Other variations may be made by composing rules from the repeated mid-point rule, trapezium rule, Simpson's or $\frac{3}{8}$ rules; Baker & Keech [3].

Properties of quadrature methods which are derived by discretizing, using the quadrature rules Q (2.3), derive from the structure of the weights ω_{ik} . It has been noted elsewhere (Baker, Makroglou & Short [4]), that the infinite array of weights $\{\omega_{ik}\}$, from the rules Q which have been constructed above, have the feature that $\{\omega_{ik}\}$ is lower triangular and can be partitioned as follows:

$$\begin{array}{c|cccccccc}
 \begin{array}{c} \underline{w}^1 \\ \sim \end{array} & & & & & & & \\
 \hline
 & \begin{array}{c} \underline{w}_P \\ \sim \end{array} & & & & & & \\
 & \underline{w}_{P-1} & \underline{w}_P & & & & & \\
 \begin{array}{c} T \\ \sim \end{array} & & & & & & & \\
 & \underline{w}_0 & \dots & \underline{w}_0 & \underline{w}_1 & \dots & \underline{w}_P & \\
 & \underline{w}_0 & & \underline{w}_0 & \underline{w}_0 & \underline{w}_1 & \underline{w}_2 & \dots \underline{w}_P \\
 & \sim & & \sim & & \sim & \sim & \sim
 \end{array} \tag{2.6}$$

where the weights of a set of starting formulae are represented by the elements of \underline{w}^1 ; $\underline{w}_0, \underline{w}_1, \dots, \underline{w}_P$ are each of a fixed order q , say, and \underline{w}_P is lower triangular. Typically (apart from initial rows) the rows of T repeat in blocks, in the cases illustrated. For the Gregory rules illustrated in Example 2.2 the value of q is 1, but otherwise this array gives a typical example of (2.6).

Example 2.4

The 'Kobayasi' weights of Example 2.3(a), corresponding to the repeated Simpson's rule and the trapezium rule, are such that, in the array, $P = 1$, $q = 2$, $\underline{w}^1 = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, $\underline{w}_0 = \begin{bmatrix} 2/3 & 4/3 \\ 2/3 & 4/3 \end{bmatrix}$,

$$\underline{w}_1 = \begin{bmatrix} 1/3 & 0 \\ 5/6 & \frac{1}{2} \end{bmatrix}; \text{ if } \underline{w} = \begin{bmatrix} 1/3 & 4/3 \\ 1/3 & 4/3 \end{bmatrix}, \quad T = \begin{bmatrix} \underline{w} \\ \sim \\ \underline{w} \\ \sim \\ \vdots \end{bmatrix}.$$

We may also consider numerical methods for the Volterra equation of the first kind

$$\int_0^x H(x,y,f(y))dy = g(x), \quad 0 \leq x \leq X \quad (2.7)$$

Use of the rules Q produces a system of equations

$$h \sum_{k=0}^i \omega_{ik} H(ih, kh, \tilde{f}_k) = g(ih), \quad i=1,2,3,\dots,N; \quad Nh = x \quad (2.8)$$

We have indicated, previously, that the Volterra equation of the first kind (2.7) may be ill-posed and that, in consequence, difficulties may be expected when solving such equations numerically. Thus, quadrature rules Q , defined in (2.3) which yield convergent methods for second kind equations may not generate convergent quadrature methods for first kind equations. We observe that the system of equations (2.8) is different in character from the system (2.4). In general, if $\omega_{i0} \neq 0$, we require, in addition to (2.8) an equation determining $f(0)$. Otherwise, if $\omega_{i0} = 0$ ($i = 1, 2, \dots, N$) then (2.8) is a system of N equations in $(N+1)$ unknowns and cannot be solved. It is also possible to produce a numerical scheme, by an appropriate choice of weights, which does not rely on all the equations in system (2.8).

Example 2.5.

Consider the repeated mid-point rule for even values of i , $i = 2j$, with weights ω_{ik} :

$k =$	0	1	2	3	4	$i-1,$	$i = 2j$	
$\omega_{2j,k}$	0	2	0	2		2	0	(2.9)

where i is odd ($i = 2j+1$) the mid-point rules must be supplemented by another rule. For convenience, we use the trapezium rule as an 'end' correction:

$k =$	0	1	2	3	4		$i-2$	$i-1$	$i=2j+1$	
$\omega_{2j+1,k}$	0	2	0	2			2	$\frac{1}{2}$	$\frac{1}{2}$	(2.10)

The solution of (2.4) by this 'Kobayasi' rule requires the use of the weights in both (2.9) and (2.10). Whereas, from systems of equations using only the weights in (2.9), it is possible to compute $\tilde{f}(h)$, $\tilde{f}(3h)$, $\tilde{f}(5h)$... in step-by-step fashion. The system of equations (2.8) with $i = 2s$ becomes

$$2hH(2sh, (2s-1)h, \tilde{f}((2s-1)h)) = g(2sh) - 2h \sum_{k=1}^{s-1} H(2sh, (2k-1)h, \tilde{f}((2k-1)h))$$

since $\omega_{2s,0} = \omega_{2s,2} = \dots = \omega_{2s,2s} = 0$ and we observe that a starting value $\tilde{f}(0)$ is not required here.

Now we turn our attention to the linear case where $H(x,y,f(y)) = K(x,y)f(y)$ and (2.7) becomes

$$\int_0^x K(x,y)f(y)dy = g(x), \quad 0 \leq x \leq X. \quad (2.11)$$

If $K(x,y)$ is continuous for $0 \leq y \leq x \leq X$ then, setting $x = 0$ in (2.11), we see that if there is a continuous solution $f(x)$ then $g(0) = 0$. In §1.3 we observed that, provided $\frac{\partial K}{\partial x}(x,y) = K_x(x,y)$ is continuous for $y \leq x$ and $g'(x)$ is continuous, we obtain

$$K(x,x) f(x) + \int_0^x K_x(x,y) f(y)dy = g'(x). \quad (2.12)$$

Thus, if $K(x,x) \neq 0$ we can endeavour to solve, numerically, the equivalent equation of the second kind

$$f(x) + \int_0^x K_1(x,y) f(y)dy = g_1(x),$$

where $K_1(x,y) = \frac{K_x(x,y)}{K(x,x)}$, $g_1(x) = \frac{g'(x)}{K(x,x)}$ and $f(0) = \frac{g'(0)}{K(0,0)}$.

If $K(x,x)$ vanishes everywhere in $0 \leq x \leq X$ then (2.12) reduces to the form (2.11) and the process of attempting to find an equation of the second kind can be repeated. This process would eventually succeed if $[(\frac{\partial}{\partial x})^r K(x,y)]_{y=x} \neq 0$ for $0 \leq x \leq X$ and some r . In practice, there may be some difficulty in evaluating the derivatives required and the technique may in any case be of little help if r is large.

Numerical methods for the Volterra integro-differential equation (1.27a,b) may be derived by applying methods for ordinary differential equations to (1.27a,b) in which the integral term is replaced by numerical quadrature. Such methods have been discussed by Linz[24], Brunner & Lambert [13] and Tavernini[32]. In addition, in §1.5.2, we have shown that the integro-differential equation may be written as a coupled pair of integral equations (1.29). Thus, techniques for the numerical solution of integral equations may be adapted to treat the system (1.29). Additional details may be found in Baker[1]. These two approaches to the development of numerical methods for integro-differential equations are not mutually exclusive. See, for example, the further comment below in §2.2.1.

2.2.1 Reducible Quadrature Methods.

If, in equation (2.1), H is independent of x and g is constant then (2.1) has the form

$$f(x) = f_0 + \int_0^x H(y, f(y)) dy \quad (2.13)$$

This is equivalent to the initial value problem

$$f'(x) = H(x, f(x)), \quad f(0) = f_0 \quad (2.14)$$

Thus, the numerical method (2.4) could be regarded as a technique for solving (2.14) albeit an unconventional one. This insight suggests that we should investigate under what conditions a numerical method for (2.13) may be regarded as a direct method for solving (2.14). We shall see that the requirement is that the quadrature weights shall display a particular structure.

Example 2.6.

The repeated trapezium rule has weights such that $\omega_{i0} = \omega_{ii} = \frac{1}{2}$ and $\omega_{ik} = 1$ for $k = 1$ (1) $i-1$. We apply this rule as a direct quadrature method to (2.13) to obtain

$$\tilde{f}_i = f_0 + h \sum_{k=0}^i \omega_{ik} H(kh, \tilde{f}_k) \quad (2.15)$$

Differencing successive equations (2.15), and using the structure of the quadrature weights $\omega_{i+1,k} = \omega_{ik}$ $k = 0(1)i-1$, yields

$$\tilde{f}_{i+1} = \tilde{f}_i + \frac{h}{2} [H(ih, \tilde{f}_i) + H((i+1)h, \tilde{f}_{i+1})] \quad (2.16)$$

We observe that (2.16) is the trapezium rule applied directly to (2.14).

We denote by

$$\rho(\mu) := \sum_{i=0}^m \alpha_i \mu^{m-i}, \quad \sigma(\mu) := \sum_{i=0}^m \beta_i \mu^{m-i} \quad (2.17)$$

the first and second characteristic polynomials of a consistent zero-stable linear m -step multistep method (ρ, σ) for 2.14 (Lambert [23]), and we ask under what conditions equations of the form (2.15) reduce to the following equations

$$\sum_{i=0}^m \alpha_i \tilde{f}_{n-i} = h \sum_{i=0}^m \beta_i H((n-i)h, \tilde{f}_{n-i}) \quad (2.18)$$

which result from the application of the (ρ, σ) method to (2.14)

From (2.15) and (2.18) we obtain, for some $n \geq n_0$,

$$\sum_{i=0}^m \alpha_i \tilde{f}_{n-i} = \sum_{i=0}^m \alpha_i f_0 + h \sum_{i=0}^m \alpha_i \sum_{j=0}^{n-i} \omega_{n-i,j} H(jh, \tilde{f}_j) \quad (2.19)$$

On writing $\omega_{ik} = 0$ for $i < k$, (2.19) can be expressed as

$$\sum_{i=0}^m \alpha_i \tilde{f}_{n-i} = \sum_{i=0}^m \alpha_i f_0 + h \sum_{i=0}^m \alpha_i \sum_{j \geq 0} \omega_{n-i,j} H(jh, \tilde{f}_j) \quad (2.20)$$

from which we derive

$$\sum_{i=0}^m \alpha_i \tilde{f}_{n-i} = \sum_{i=0}^m \alpha_i f_0 + h \sum_{j \geq 0} \sum_{i=0}^m \alpha_i \omega_{n-i,j} H(jh, \tilde{f}_j) \quad (2.21)$$

Thus, allowing for a variation in starting procedures, the equivalence of (2.15) and (2.18) is preserved if the weights of the rules Q in (2.3) are (ρ, σ) -reducible as follows:

Definition 2.2

Assume (ρ, σ) defines a zero-stable consistent linear multistep formula. The rules Q are (ρ, σ) -reducible if and only if for some $n_0 > 0$

$$\sum_{\ell \geq 0} \alpha_{\ell} \omega_{n-\ell, j} = \beta_{n-j} \quad (n \geq n_0) \quad (2.22)$$

where $\omega_{kj} = 0$ for $k < j$, and $\alpha_{\ell}, \beta_{\ell} = 0$ for $\ell \notin \{0, 1, \dots, m\}$.

The Gregory rules (Baker [2], Baker & Keech [3]) reduce to Adams-Moulton multistep formulae; other reducible rules can be generated from (ρ, σ) (see Wolkenfelt [33], [37] and Matthys [26]).

Example 2.7

The repeated trapezium rule discussed in Example 2.6 with $\omega_{i0} = \omega_{ii} = \frac{1}{2}$, $\omega_{ik} = 1$, $k = 1(1)i-1$, is such that $\alpha_0 = -\alpha_1 = 1$, $\beta_0 = \beta_1 = \frac{1}{2}$, $m = 1$.

Returning briefly to our earlier comment in §2.2 on numerical methods for integro-differential equations we observe that a linear multi-step method applied to (1.27a,b) with the integral terms discretized by the quadrature rules Q may be equivalent to a quadrature method applied to (1.29).

2.2.2. Cyclically Reducible Rules

Rules such as the repeated mid-point rule and repeated mid-point with trapezium rules of Example 2.5 and the 'Kobayasi' rules of Example 2.3 are not reducible to linear multi-step methods but to q-cyclic linear multi-step methods (Stetter [31]) in which characteristic polynomials are defined:

$$\rho^{(\nu)}(\mu) = \sum_{i=0}^m \alpha_i^{(\nu)} \mu^{m-i}, \quad \sigma^{(\nu)}(\mu) = \sum_{i=0}^m \beta_i^{(\nu)} \mu^{m-i} \quad (2.23)$$

for $\nu = 0, 1, \dots, q-1$. Thus, we arrive at the following:

Definition 2.3

The rules Q are $\{\rho^{(\nu)}, \sigma^{(\nu)}\}_{\nu=0}^{q-1}$ -cyclically reducible if and only if for some $n_0 > 0$ and $n_1 \geq 0$

$$\sum_{\ell \geq 0} \alpha_{\ell}^{(\nu)} \omega_{n-\ell, j} = \beta_{n-j}^{(\nu)} \quad n \geq n_0; \quad \nu \in \{0, 1, \dots, q-1\} \quad (2.24)$$

where $\nu \equiv \{n - n_1\} \bmod q$.

Example 2.8

Consider the weights in Example 2.3 (a) (Repeated Simpson rule with the trapezium rule as an 'end' correction). The rules which they produce may reduce to more than one set of 2-cyclic formulae (Baker & Wilkinson [7]).

$$\begin{aligned} \text{We may define } \alpha_{\ell}^{(\nu)} &= \beta_{\ell}^{(\nu)} = 0 & \alpha_0^{(0)} &= -\alpha_1^{(0)} = 1; \\ \beta_0^{(0)} &= \frac{1}{3}; \beta_1^{(0)} = \frac{5}{6}; \beta_2^{(0)} = -\frac{1}{6}; & \alpha_0^{(1)} &= -\alpha_1^{(1)} = 1; \\ \beta_0^{(1)} &= \beta_1^{(1)} = \frac{1}{2}; & \alpha_{\ell}^{(\nu)} &= \beta_{\ell}^{(\nu)} = 0 \text{ otherwise.} \end{aligned}$$

Alternatively,

$$\begin{aligned} \alpha_0^{(0)} &= -\alpha_2^{(0)} = 1; \beta_0^{(0)} = \beta_2^{(0)} = \frac{1}{3}; \beta_1^{(0)} = \frac{4}{3}; \\ \alpha_0^{(1)} &= -\alpha_1^{(1)} = 1; \beta_0^{(1)} = \beta_1^{(1)} = \frac{1}{2}; \alpha_{\ell}^{(\nu)} = \beta_{\ell}^{(\nu)} = 0 \text{ otherwise.} \end{aligned}$$

2.2.3 Construction of quadrature weights

We observe that we may select a linear multistep method for (2.14) and then determine the weights ω_{ik} of the quadrature rules for a numerical method for (2.13).

From (2.22) it can be shown that (see Wolkenfelt [38]) the weights $w_{n,j}$ may be arranged in a matrix of the form

$$Q = \left[\begin{array}{ccc|ccc} w_{0,0} & \dots & w_{0,m-1} & & & \\ \vdots & & \vdots & & & \\ w_{m-1,0} & \dots & w_{m-1,m-1} & & & \\ \hline w_{m,0} & \dots & w_{m,m-1} & w_{m,m} & & \\ \vdots & & \vdots & \vdots & \ddots & \\ w_{n,0} & \dots & w_{n,m-1} & w_{n,m} & \dots & w_{n,n} \end{array} \right] = \left[\begin{array}{c|c} \tilde{w}_m^1 & \tilde{0} \\ \hline \tilde{T}_m & \tilde{\Omega}_n \end{array} \right] \quad (2.27)$$

in which $w_{n,j} = \omega_{n-j}$ for $n-j \geq 0$, $j \geq m$.

weights $\omega_{n,j}$ have a repetition factor r . The definition of the repetition factor r was originally given by Linz [24]. For Examples 2.1, 2.3a and 2.6 the repetition factor is unity, whilst for example 2.3b it is two.

Wolkenfelt [35] has derived the more general notion of the asymptotic repetition factor as follows:

Definition 2.4(b)

The weights ω_{nj} in (2.3) are said to have an asymptotic repetition factor r if r is the smallest positive integer such that

$$\lim_{n \rightarrow \infty} (\omega_{n+r,j} - \omega_{n,j}) = 0 \quad \text{for all } j, \quad j_1 \leq j \leq n-j_2 \quad (2.31)$$

We shall refer again to the concept of repetition factor in discussing numerical stability of direct quadrature methods (Wolkenfelt [35]) and of methods derived by imbedding techniques; see Wolkenfelt, van der Houwen & Baker [39].

2.3 Runge-Kutta-Type Methods for Volterra equations.

Methods of Runge-Kutta-type for the Volterra equation of the second kind

$$f(x) - \int_0^x H(x,y,f(y))dy = g(x), \quad x \geq 0$$

are various analogues of the methods of Runge-Kutta-type for the solution of initial-value problems in ordinary differential equations of the form $f'(x) = F(x,f(x))$. Two approaches in the development of these methods are those of Pouzet [30] and Beltyukov [8]. Pouzet derives his formula for use with integral equations from corresponding Runge-Kutta methods for ordinary differential equations and, therefore, the values θ and A (see below) are available from these latter methods. Beltyukov uses a more direct approach, extended by Brunner, Hairer, Norsett [12].

2.3.1. Classical Extended Runge-Kutta Methods

The approach of Pouzet is illustrated in the following way.

A classical Runge-Kutta method for an ordinary differential equation of the form

$$f'(x) = F(x, f(x)), \quad x \geq 0, \quad f(0) = f_0 \quad (2.32)$$

may be defined by a tableau of the form

$$[\theta | A] = \begin{array}{c|cccccc} \theta_0 & A_{00} & A_{01} & \dots & A_{0,p-1} & A_{0p} \\ \theta_1 & A_{10} & A_{11} & \dots & A_{1,p-1} & A_{1p} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \theta_{p-1} & A_{p-1,0} & A_{p-1,1} & \dots & A_{p-1,p-1} & A_{p-1,p} \\ \hline \theta_p=1 & A_{p,0} & A_{p,1} & \dots & A_{p,p-1} & A_{pp} \end{array} \quad (2.33)$$

It is sometimes convenient to write b_s for $A_{p,s}$.

and the formulae

$$\tilde{f}_{n+1} = \tilde{f}_n + \sum_s A_{ps} K_s^{(n)}, \quad (2.34)$$

$$K_r^{(n)} = hF(nh + \theta_r h, \tilde{f}_n + \sum_s A_{rs} K_s^{(n)}) \quad (r=0,1,\dots,p),$$

$$\tilde{f}_0 = f_0.$$

In general, summations over s are for $s = 0, 1, \dots, p$. (For an explicit method $A_{rs} = 0$ if $r \geq s$, so the summation can be regarded as running for $s = 0, 1, \dots, p-1$ in the second equation of (2.34).)

If $F(x, f(x)) = \lambda f(x)$ and $\tilde{f}_n \rightarrow 0$ as $n \rightarrow \infty$ when $\text{Re}(\lambda) < 0$ the method is called A-stable. For details and related terminology, see Lambert [23]. An alternative formula may be achieved if we define

$$f_{n,r} = \tilde{f}_n + \sum_s A_{rs} K_s^{(n)}, \quad (2.35)$$

where we note that

$$f_{n-1,p} = \tilde{f}_n \cong f(nh). \quad (2.36)$$

Thus, the Runge-Kutta method may be defined by the formulae

$$f_{n,r} = f_{n-1,p} + h \sum_s A_{rs} F(nh + \theta_s h, f_{n,s}), \quad (r = 0, 1, \dots, p) \quad (2.37)$$

in which we take $f_{n,r}$ as defined in (2.35) to be an approximation

to $f(nh + \theta_r h)$ and $f_{-1,p} = f_0$. As the rows of the tableau (2.33) may be regarded as defining a family of rules

$$\int_0^{\theta_r h} \varphi(y) dy \approx \sum_{s=0}^r A_{rs} \varphi(\theta_s h), \quad (r=0,1,\dots,p) \quad (2.38)$$

we see that (2.37) may be regarded as a discretization of

$$f(nh + \theta_r h) = f(nh) + \int_{nh}^{nh+\theta_r h} F(y, f(y)) dy. \quad (2.39)$$

We have observed that linear multi-step rules in methods for ordinary differential equations are related to quadrature methods for integral equations. Hence, it is natural to ask whether Runge-Kutta methods for ordinary differential equations have an association with methods for integral equations.

Summing (2.37) over n we obtain

$$f_{n,r} = f_0 + h \sum_{k=0}^{n-1} \left\{ \sum_s A_{ps} F(kh + \theta_s h, f_{k,s}) \right\} + h \sum_s A_{rs} F(nh + \theta_s h, f_{n,s}),$$

$$(r = 0, 1, \dots, p) \quad (2.40)$$

which we may regard as a discretization of the integral equation

$$f(x) - \int_0^x F(y, f(y)) dy = f_0. \quad (2.41)$$

For the general Volterra equation of the second kind

$$f(x) - \int_0^x H(x, y, f(y)) dy = g(x)$$

we obtain

$$f_{n,r} = h \sum_{k=0}^{n-1} \sum_s A_{ps} H(nh + \theta_r h, kh + \theta_s h, f_{k,s}) + h \sum_s A_{rs} H(nh + \theta_r h, nh + \theta_s h, f_{n,s})$$

$$+ g(nh + \theta_r h), \quad (r = 0, 1, \dots, p) \quad (2.42)$$

We may regard this approach as one in which we consider the integral on $[0, nh]$ and on $[nh, nh + \theta_r h]$ separately. For the discretization over the first interval $[0, nh]$ the n -times repeated version of (2.37) with $r = p$ provides the first term on the right-hand side of (2.42), the rule (2.37) with appropriate r provides the second sum in (2.42).

This method for integral equations, using as it does the Runge-Kutta tableau for ordinary differential equations, is defined to be the (classical) extended Runge-Kutta method.

Example 2.9

An analogue of the fourth order Runge-Kutta method for ordinary differential equations is given by choosing $p = 4$ and the tableau

0				
$\frac{1}{2}$	$\frac{1}{2}$			
$\frac{1}{2}$	0	$\frac{1}{2}$		
1	0	0	1	
1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

where values omitted from the tableau are zero.

The approach of Beltyukov [8] is frequently discussed in relation to the Volterra equation in its canonical form, namely,

$$\varphi(x) = \int_0^x G(x, y, \varphi(y)) dy \quad (2.43)$$

The objective is to devise an approximate value $\varphi(h)$, where $\tilde{\varphi}(h) = \varphi(h) + O(h^p)$, by a direct approach displaying the basic philosophy of Runge-Kutta methods in the process. By direct differentiation of (2.43) and a Taylor series approximation Beltyukov develops formulae of the type (Baker [2, p. 862])

$$\tilde{\varphi}(h) = \sum_s A_{ps} K_s \quad (2.44)$$

$$K_r = hG(\alpha_r h, \theta_r h, \sum_s A_{rs} K_s). \quad (r = 0, 1, \dots, p-1)$$

which are linked to an augmented Runge-Kutta tableau $[\alpha | \theta | A]$.

Example 2.10 (Baker [2, p.864])

In the case $p = 3$ a system of 13 equations in 12 unknowns is obtained for the determination of the parameters α, θ, A . One solution is given by the tableau

$$\left[\begin{array}{c|c|c} \alpha & \theta & A \\ \hline \sim & \sim & \sim \end{array} \right] =$$

$$\left[\begin{array}{ccc|cccc} 1 & . & 0 & 0 & 0 & 0 & 0 \\ & . & . & . & . & . & . \\ \frac{1}{2} & . & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ & . & . & . & . & . & . \\ 1 & . & 2/3 & 2/9 & 4/9 & 0 & 0 \\ \hline 1 & . & 1 & \frac{1}{4} & 0 & 3/4 & 0 \end{array} \right]$$

2.3.2. Mixed Quadrature - Runge-Kutta Methods.

Consider

$$f(x) - \int_0^x H(x, y, f(y)) dy = g(x), \quad x \geq 0. \quad (2.45)$$

We may derive

$$f(x+nh) - \int_0^x H(x+nh, y+nh, f(y+nh)) dy - \int_0^{nh} H(x+nh, y, f(y)) dy = g(x+nh). \quad (2.46)$$

With

$$f_n^*(x) := f(x+nh),$$

$$g_n^*(x) := \int_0^{nh} H(x+nh, y, f(y)) dy + g(x+nh) \quad (2.47)$$

we may write (2.46) as

$$f_n^*(x) - g_n^*(x) = \int_0^x H(x+nh, y+nh, f_n^*(y)) dy \quad (2.48)$$

in which the problem of approximating $f((n+1)h)$ becomes that of approximating $f_n^*(h)$.

Following the Pouzet approach, we set $r = p$ in (2.42) and obtain the formula for discretizing (2.48) when $g_n^*(x)$ is approximated as follows:

$$\tilde{g}_n^*(x) = h \sum_{k=0}^{n-1} \sum_s A_{ps} H(x+nh, kh+\theta_s h, f_{k,s}) + g(x+nh)$$

Alternatively, following the Beltyukov approach (which involves the treatment of (2.43)), we compare (2.48) with (2.43) and see that we may set

$$\varphi(x) = f_n^*(x) - g_n^*(x) \quad \text{and} \quad G(x, y, v) = H(x+nh, y+nh, v+g_n^*(y)).$$

Hence, we may use (2.44) as a numerical method for an approximate solution of (2.45) provided an approximation for $g_n^*(x)$ can be found. Beltyukov mentions the Gregory formulae as a possibility.

We have noted that Pouzet and Beltyukov offer alternative approximations to the term $g_n^*(x)$. The family of quadrature rules

$$\int_0^{nh} \varphi(y) dy \approx h \sum_{k=0}^n \omega_{nk} \varphi(kh), \quad (n=1,2,\dots; h > 0) \quad (2.49)$$

provides an alternative approach to extended methods of Pouzet in approximating over the interval $[0, nh]$. This yields the mixed-quadrature-Runge-Kutta methods for Volterra equations of the second kind, of the form

$$f_{n,r} = h \sum_{k=0}^n \omega_{nk} H(nh + \theta_r h, kh, f_{k-1,p}) + h \sum_s A_{rs} H(nh + \theta_r h, nh + \theta_s h, f_{n,s}) + g(nh + \theta_r h), \quad (r = 0, 1, \dots, p) \quad (2.50)$$

where $f_{-1,p} = f(0) = g(0)$.

Note: The scheme proposed by Beltyukov for approximating $g_n^*(x)$ is a slight variation of this mixed quadrature R-K method.

2.3.3. Extended and Mixed Runge-Kutta Methods as extensions of the Quadrature Methods.

Hairer [15] has considered the derivation of R-K Methods for Volterra equations as starting formulae with auxiliary discretizations of the lag term. In [15], he has presented the theory of extended R-K Methods for Volterra equations in a compact form. first stating a convergence theorem then constructing some R-K Methods whose stability is also investigated.

However, it is of interest to note that the quadrature methods of §2.2 and the extended and mixed R-K methods of §§2.3.1-2.3.2 respectively could be incorporated into a general analysis in which we view these methods as extensions of the quadrature methods. To achieve this we need to re-index the variables. We write

$$\tilde{f}_j = f(\tau_j) = f_{ir} \quad \text{where} \quad \tau_j = ih + \theta_r h, \quad r = 0, 1, \dots, p \quad (2.51)$$

successively for $i = 0, 1, 2, \dots$ with $j = i(p+1) + r + 1$, $\tau_0 = 0$ and $\tilde{f}_0 = f_0$. We note that $i \equiv [(j-1)/(p+1)]$, $r \equiv (j-1) \bmod(p+1)$, where $[x]$ denotes the integer part of x .

Thus, classical R-K formulae of the form (2.42) for the discretization of the Volterra equation of the second kind produce formulae of the type

$$\tilde{f}_j = h \sum_{k \geq 0} \Omega_{jk} H(\tau_j, \tau_k, \tilde{f}_k) + g(\tau_j), \quad j = 1, 2, 3, \dots \quad (2.52)$$

in which the extended R-K method is regarded as the extension of a quadrature method where the 'weights' are denoted

$$\Omega_{jk} = \Omega_{jk}(A) \quad \text{and} \quad \Omega_{jk}(A) = \begin{cases} A_{pt}, & 0 < k \leq i(p+1), \\ A_{rt}, & i(p+1) < k \leq (i+1)(p+1), \\ 0 & \text{otherwise,} \end{cases} \quad (2.53)$$

where $t \equiv (k-1) \bmod(p+1)$, $r \equiv (j-1) \bmod(p+1)$

For the mixed-quadrature - R-K methods of the form (2.50) we have formulae of the type (2.52) in which the 'weights' are denoted

$$\Omega_{jk} = \Omega_{jk}[Q, A]$$

and

$$\Omega_{jk}[Q, A] = \begin{cases} \omega_{im}, & k = m(p+1), \quad m \leq i \\ A_{rt}, & i(p+1) < k \leq (i+1)(p+1), \\ 0 & \text{otherwise;} \end{cases} \quad (2.54)$$

r, t being defined as for (2.53), i being $[j-1]/(p+1)$.

Finally, the quadrature methods (2.4) fit into the framework (2.52) in which we set

$$\Omega_{jk} = \Omega_{jk}(Q) \equiv \omega_{jk}, \quad \tau_j = jh. \quad (2.55)$$

The preceding framework is sometimes convenient although high-order accuracy now appears as a superconvergence phenomenon which may be difficult to prove.

2.3.4 Some variants of extended Runge-Kutta and Mixed Quadrature - Runge-Kutta Methods.

One variant of the procedures already described in §2.3.3 is what we shall call here the "economized version". In the case when $\theta_0 = 0$ either of the schemes already described yield $\tau_{i(p+1)} = \tau_{i(p+1)+1} = ih$ (since $\theta_p = 1$). Thus, effectively, we have two approximations at the point ih . We may achieve some apparent economy of effort if we modify the methods described by setting $\tilde{f}(ih) = \tilde{f}_{i(p+1)} = \tilde{f}_{i(p+1)+1}$ rather than compute the new value. In the case of extended methods with $\theta_0 = 0$ and $A_{0r} = 0$ ($r = 0, 1, \dots, p$) (such as an explicit method) the original method is already an economized version.

A further class of methods arises on replacing $[A_{p0}, A_{p1}, \dots, A_{pp}]^T$ in the R-K tableau (2.33) by $\underline{b} = [b_0, b_1, \dots, b_{p-1}, b_p]^T$ where \underline{b}^T is a row of quadrature weights associated with abscissae $\theta_0, \dots, \theta_p$. For example, the row \underline{b}^T may be a row of another R-K method associated with the same points $\theta_0, \theta_1, \dots, \theta_p$ of the R-K tableau (2.33). Such a method is proposed by Pouzet [30]. The 'weights' (2.53) are then denoted

$$\Omega_{jk} = \Omega_{jk}(\underline{b}, A)$$

$$\text{and } \Omega_{jk}(\underline{b}, A) = \begin{cases} b_t, & 0 < k \leq i(p+1), \\ A_{rt}, & i(p+1) < k \leq (i+1)(p+1), \\ 0 & \text{otherwise; } t \equiv (k-1) \bmod (p+1). \end{cases} \quad (2.56)$$

We observe that methods of Beltyukov type (Baker [2]) reduce to economised mixed methods when the kernel $G(x, y, f(y))$ is independent of x .

2.3.5 Structure

We noted in §2.2 that a feature of the rules $\Omega(Q)$ was that the infinite array $\{\omega_{ik}\}$ is lower triangular and can be partitioned into a form given by (2.6).

In the extended R-K method the weights $\Omega_{jk}(\underline{b}, A)$ in (2.56) have the

form, when partitioned

$$\begin{array}{ccccccc}
 A & & & & & & \\
 \sim & & & & & & \\
 B & A & & & & & \\
 \sim & \sim & & & & & \\
 B & B & A & & & & \\
 \sim & \sim & \sim & & & & \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 B & B & \cdot & \cdot & B & A & \\
 \sim & \sim & & & \sim & \sim &
 \end{array}
 \quad \text{where } B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} b^T
 \quad (2.57)$$

With the obvious parallel in structure of the quadrature rules we are prompted to give the following definition.

Definition 2.5

A block-lower triangular array of weights (Ω_{jk}) is block-reducible or $(A_\ell, B_\ell)_0^m$ -reducible if it may be partitioned into square sub-matrices $V_{n\ell}$ of order q , namely,

$$(\Omega_{jk}) = \begin{bmatrix} V_{0,0} & & & & \\ \sim & & & & \\ \vdots & & & & \\ V_{\ell,0} & V_{\ell,1} & \dots & & V_{\ell,\ell} \\ \sim & \sim & & & \sim \\ V_{\ell+1,0} & V_{\ell+1,1} & \dots & & V_{\ell+1,\ell} & V_{\ell+1,\ell+1} \\ \sim & \sim & & & \sim & \sim \end{bmatrix} \quad (2.58)$$

such that, for some n_0 ,

$$\sum_{\ell=0}^m A_\ell \cdot V_{n-\ell, j} = B_{n-j}, \quad n \geq n_0 \quad (2.59)$$

where $A_\nu = B_\nu = 0$ for $\nu \notin \{0, 1, 2, \dots, m\}$ and $V_{n,\ell} = 0$ if $\ell > n$.

We usually ask that $\sum_{\ell=0}^m A_\ell \epsilon = 0$ where $\epsilon = [1, 1, \dots, 1]^T \in \mathbb{R}^q$.

If $A_0 = I$, $A_1 = -I$, $A_\ell = 0$ otherwise the rules are called simply block-reducible.

When $\Omega_{jk} = \Omega_{jk}(Q)$, $q = 1$ and $A_\ell = \alpha_\ell$, $B_\ell = \beta_\ell$ with $V_{n,\ell} = \omega_{n\ell}$ the rules are (ρ, σ) -reducible and we see that (2.59) correspond to generalisations of (2.22) in Definition 2.2.

A subset of the block-reducible rules are those for which (2.58) assumes a special form. Thus, the weights in the extended R-K methods and in the quadrature methods as well as in the mixed-quadrature - R-K methods can be partitioned (Baker & Wilkinson [7, eq.(2.9)] as shown in (2.60) (or a similar form indicated by (2.6)):

$$\begin{bmatrix} \tilde{W}' \\ \sim \\ \tilde{W}'' & \tilde{W}_P \\ \sim & \sim \\ \tilde{W} & \tilde{W}_0 & \tilde{W}_1 & \dots & \tilde{W}_P \\ \sim & \sim & \sim & & \sim \\ \tilde{W} & \tilde{W}_0 & \tilde{W}_0 & \dots & \tilde{W}_0 & \tilde{W}_1 & \tilde{W}_2 & \dots & \tilde{W}_P \\ \sim & \sim & \sim & & \sim & \sim & \sim & & \sim \\ \tilde{W} & \tilde{W}_0 & \tilde{W}_0 & \dots & \tilde{W}_0 & \tilde{W}_0 & \tilde{W}_1 & \dots & \tilde{W}_{P-1} & \tilde{W}_P \\ \sim & \sim & \sim & & \sim & \sim & \sim & & \sim & \sim \end{bmatrix} \quad (2.60)$$

(\tilde{W}' , \tilde{W}'' , \tilde{W} may be rectangular or square).

The array in (2.60) corresponds to (2.59), where in (2.59), we have taken $\tilde{V}_{\ell,0} = \tilde{W}$, $\ell \geq n_0$; $\tilde{V}_{\ell,j} = \tilde{W}_{P-\ell+j}$ for $j \geq 1$, and where $\tilde{W}_{P+r=0}, \tilde{W}_{-r} = \tilde{W}_0$, for $r > 0$. Such rules are simply block-reducible.

Example 2.11

When the weights have the structure (2.60), define $\tilde{A}_0 = \tilde{I}$ and $\tilde{A}_1 = -\tilde{I}$ and $\tilde{A}_\ell = 0$ otherwise; $m = P$. Then $\tilde{B}_0 = \tilde{W}_P$, $\tilde{B}_1 = \tilde{W}_{P-1} - \tilde{W}_P$, \dots $\tilde{B}_P = \tilde{W}_0 - \tilde{W}_1$ and $\sum_{\ell=0}^m \tilde{A}_\ell = 0$.

Example 2.12

Consider the extended R-K methods $\Omega_{jk}(\underline{A})$ of (2.53). They may be derived from (2.60) in which we take $P = 1$, $\tilde{W}_1 = \underline{A}$ and $\tilde{W}_0 = \underline{W}$ which is the matrix each of whose rows is the last row of \underline{A} . To derive $\Omega_{jk}(\underline{b}, \underline{A})$ in (2.56) take $P = 1$, $\tilde{W}_1 = \underline{A}$ and $\tilde{W}_0 = \underline{B}$ in (2.60), where each of the rows of \underline{B} is \underline{b}^T .

Example 2.13

Consider the Kobayasi quadrature rules of Example (2.3a). In (2.60) take $P = 1$ and

$$\tilde{W} = \begin{bmatrix} 1/3 & 4/3 \\ 1/3 & 4/3 \end{bmatrix}, \quad \tilde{W}_0 = \begin{bmatrix} 2/3 & 4/3 \\ 2/3 & 4/3 \end{bmatrix}, \quad \tilde{W}_1 = \begin{bmatrix} 1/3 & 0 \\ 5/6 & 1/2 \end{bmatrix}.$$

If the quadrature rules Q are cyclically reducible (cf. Definition 2.3), then $\Omega_{jk}(Q)$ are block reducible with

$$\tilde{A}_0 = \begin{bmatrix} \alpha_0^{(0)} & & & & \\ \alpha_1^{(1)} & \alpha_0^{(1)} & & & \\ \vdots & \vdots & \ddots & & \\ \vdots & \vdots & & \ddots & \\ \alpha_{q-1}^{(q-1)} & \alpha_{q-2}^{(q-1)} & \dots & \dots & \alpha_0^{(q-1)} \end{bmatrix}$$

$$\tilde{A}_1 = \begin{bmatrix} \alpha_q^{(0)} & \dots & \alpha_1^{(0)} \\ \alpha_{q+1}^{(1)} & \dots & \alpha_2^{(1)} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \alpha_{2q-1}^{(q-1)} & \dots & \alpha_q^{(q-1)} \end{bmatrix}$$

and so on and likewise for $\tilde{B}_0, \tilde{B}_1, \dots$ (Stetter [31, p.218]).

Example 2.14

Consider the weights in Example (2.3a) which have also provided the weights for Example (2.8). They can be partitioned as (2.60) and treated as in Example (2.13). Alternatively, we may define

$$\tilde{A}_\ell = \tilde{B}_\ell = 0 \quad \text{if } \ell > 1 \quad \text{and}$$

$$\tilde{A}_0 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad \tilde{A}_1 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_0 = \begin{bmatrix} 1/3 & 0 \\ 1/2 & 1/2 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} -1/6 & 5/6 \\ 0 & 0 \end{bmatrix}.$$

so that $\tilde{A}_0 \tilde{W}_0 + \tilde{A}_1 \tilde{W}_1 = \tilde{B}_1$, $\tilde{A}_0 \tilde{W}_1 = \tilde{B}_0$. Yet again, another formulation leads to

$$\tilde{A}_0 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad \tilde{A}_1 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_0 = \begin{bmatrix} 1/3 & 0 \\ 1/2 & 1/2 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} 1/3 & 4/3 \\ 0 & 0 \end{bmatrix}.$$

Each formulation corresponds to a recognisable 2-cyclic method.

If the rules Q are (ρ, σ) -reducible or cyclically reducible then the weights $\Omega_{jk}[Q, A]$ are block reducible.

Example 2.15

Consider the mixed quadrature R-K method employing the repeated trapezium rule and the R-K tableau $\frac{1}{2} \mid \frac{1}{2} \ 0$
1 \mid 1 0. The array $\Omega_{jk}(Q, A)$ may be partitioned

0	$\frac{1}{2}$	0				
0	1	0				
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0		
$\frac{1}{2}$	0	$\frac{1}{2}$	1	0		
$\frac{1}{2}$	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0
$\frac{1}{2}$	0	1	0	$\frac{1}{2}$	1	0
$\frac{1}{2}$	0	1	0	1	0	$\frac{1}{2}$
$\frac{1}{2}$	0	1	0	1	0	$\frac{1}{2}$

in which $\tilde{W}_0 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $\tilde{W}_1 = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$, $\tilde{W}_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & 0 \end{bmatrix}$.

We may take $\tilde{A}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\tilde{A}_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\tilde{B}_0 = \tilde{W}_2$, $\tilde{B}_1 = \tilde{W}_1 - \tilde{W}_2$, $\tilde{B}_2 = \tilde{W}_0 - \tilde{W}_1$.

2.3.6. Non-singular Volterra equations of the first kind - Revisited.

We turn our attention, once again to the linear equation

$$\int_0^x K(x, y) f(y) dy = g(x). \quad (2.61)$$

We have already discussed in §1.3.1 and §2.2 the conditions under which

the equation of the first kind may be recast as an equation of the second kind. Indeed, it is tempting to adapt our quadrature methods for Volterra equations of the second kind to deal with equations of the first kind. However, it is our purpose here to refer to the work of de Hoog & Weiss [17] whose methods generally give high-order accuracy under reasonable assumptions. ^{*} Weiss chooses $0 \leq \theta_0 < \theta_1 < \dots < \theta_p = 1$ and constructs interpolating quadrature formulae using values of the integrand evaluated at $\theta_0, \theta_1, \dots, \theta_p$ of the form

$$\int_0^{\theta_r} \varphi(y) dy \approx \sum_{s=0}^p A_{rs} \varphi(\theta_s) \quad (r = 0, 1, \dots, p) \quad (2.62)$$

On discretizing (2.61) with $x = ih + \theta_r h$ we may obtain

$$h \sum_{k \geq 0} \Omega_{jk}(A) K(\tau_j, \tau_k) f_k = g(\tau_j) \quad (2.63)$$

where $j = i(p+1) + r + 1$, $\tau_j = ih + \theta_r h$ and in which $\Omega_{jk}(A)$ are the weights given in (2.53) and A is the matrix of coefficients of (2.33). The summation work does not extend beyond $k = (i+1)(p+1)$.

In the case when $\theta_0 = 0$ it is sufficient to set $r = 0, 1, \dots, p-1$ and the equation corresponding to $i = r = 0$ is replaced by the special starting value $f_0 = \frac{g'(0)}{K(0,0)}$. When $\theta_0 > 0$ no such starting value is required.

We observe that in (2.63) values of $K(x, y)$ for $y > x$ are required. (A_{rs} does not vanish for $s > r$)

Thus, a modified method is constructed of the form:

$$\sum_{0 \leq k \leq i(p+1)} h \Omega_{jk}(A) K(\tau_j, \tau_k) f_k + h \sum_{s=0}^p \theta_r A_{ps} K(\tau_j, ih + \theta_r \theta_s h) \sum_{t=0}^p \varrho_{p,t}(\theta_r \theta_s) f_{i(p+1)+t+1} = g(\tau_j) \quad (2.64)$$

The second term on the left of (2.64) arises from the use of the quadrature rule

$$\int_0^{\theta_r} \varphi(y) dy \approx \sum_{s=0}^p \theta_r A_{ps} \varphi(\theta_r \theta_s)$$

* The techniques of de Hoog & Weiss can be adapted for second kind equations.

obtained from (2.62) by a change of variables.

The values $\tilde{f}(ih+\theta_r\theta_sh)$ are obtained by polynomial interpolation to the values $f(ih+\theta_rh)$, $r = 0, 1, 2, \dots, p$ using the interpolation polynomial

$$\varrho_{p,t}(\theta) = \prod_{\substack{s \neq t \\ s=0}}^p (\theta - \theta_s) / (\theta_t - \theta_s).$$

Equations (2.63) and (2.64) may be solved as block-by-block methods for the values $f_{i(p+1)+r+1}$, ($r = 0, 1, 2, \dots, p$) simultaneously. Other block-by-block methods typified, for example, by those of Linz may be found in the literature (Baker [2, §6.7]) where we note that the different methods of defining $H(x, y, f(y))$ when $y > x$ yield the differing versions of these block methods.

2.4. Product Integration Methods

The methods described in §§2.2 and 2.3 are inadequate for Abel equations. We mention these only briefly since we shall not discuss Abel equations in the remainder of the thesis. Product integration techniques which may be employed for such equations may also be developed for kernels of Volterra equations which can be decomposed into products of weakly-singular and well-balanced kernels. Thus, we refer general equations of the form

$$\int_0^x H_1(x, y) H_2(x, y, f(y)) dy = g(x) \quad (2.65)$$

$$f(x) - \int_0^x H_1(x, y) H_2(x, y, f(y)) dy = g(x) \quad (2.66)$$

where $H_2(x, y, f(y))$ is smooth. The forms assumed by $H_1(x, y)$ (which is required to vanish for $y > x$) include $K(x-y)$, where $K(t)$ is permitted to be continuous, or weakly singular (e.g. $K(t) = 1/t^\alpha$, $0 < \alpha < 1$). When $H_1(x, y) = 1$ ($y \leq x$) we have the Volterra equations considered earlier. With $\psi(x, y) = H_2(x, y, f(y))$, the integrands in (2.65)-(2.66) have the form $H_1(x, y)\psi(x, y)$.

We may construct "generalised quadrature" rules

$$\int_0^x H_1(x,y)\varphi(x,y)dy \approx \sum_k \nu_k(x) \varphi(x,kh), \quad x \in \{ih\} \quad (2.67)$$

which are exact when $\varphi(x,y)$ is a polynomial of a certain degree in y , or when $\varphi(x,y)$ is piecewise-polynomial in y and $x \in \{ih\}$. In particular, the approximation may be constructed to be exact (a) if $\varphi(x,y)$ is linear in y in each interval $kh \leq y \leq (k+1)h$, ($k = 0, 1, 2, \dots$); (b) if $\varphi(x,y)$ is quadratic in y in each interval $2kh \leq x \leq (2k+2)h$, ($k = 0, 1, 2, \dots$) and so on.

Approximating $\varphi(ih,y)$ by a piecewise polynomial $\tilde{\varphi}(ih,y)$ agreeing with $\varphi(ih,y)$ for $y \in \{kh\}_{k \geq 0}$ and integrating $H_1(x,y)\tilde{\varphi}(x,y)$ over $[0,x]$ we may choose $\{\nu_k(ih)\}$ so that

$$\sum_k \nu_k(x) \varphi(x,kh) = \int_0^x H_1(x,y)\tilde{\varphi}(x,y)dy, \quad x \in \{ih\} \quad (2.68)$$

Firstly we pause to sound a note of caution. In solving a Volterra equation numerically we are concerned to examine whether or not the long time behaviour of the discretized equation reflects that of the original. In other words, we are interested in the concepts of convergence and stability (see later). Clearly, it cannot be expected that it would be possible to provide a general answer which would cover the whole class of integral equations. We should also be aware of the work of Lubich [25], for example. His approach limits consideration to methods which are (using the classical stability concepts of ordinary differential equations) strongly stable, A-stable and $A(\alpha)$ -stable and to the class of problems for which these properties are maintained. Papers published by Nevanlinna [27], [28] and [29] adopt a similar approach.

2.5 Bownds' method

Bownds' method [9], for the treatment of (2.1) is based upon the approximation of $H(x,y,f(y))$ in (2.1) by a finite sum

$$\sum_{r=1}^R X_r(x) Y_r(y, f(y)).$$

(The latter is a finitely separable kernel approximating H .) If we consider the equation

$$f(x) - \int_0^x \sum_r X_r(x) Y_r(y, f(y)) dy = g(x), \quad (2.69)$$

when the sum is a finite one, we find that the equation (2.69) can be reduced to the solution of an initial-value problem of a system of differential equations. Indeed, from (2.69)

$$f(x) = g(x) + \sum_r a_r(x) X_r(x) \quad (2.70)$$

where

$$a_r(x) = \int_0^x Y_r(y, f(y)) dy,$$

that is

$$a'_r(x) = Y_r(x, f(x))$$

or

$$a'_r(x) = Y_r(x, g(x) + \sum_s a_s(x) X_s(x)). \quad (2.71)$$

We have $a_r(0) = 0$. Solution of the system of differential equations (2.71) provides $f(x)$ from (2.70). Thus, Bownds advocates the construction of appropriate finitely-separable approximations to H and the solution of (2.71) by automatic methods for initial-value problems. It may be noted that the application of a (ρ, σ) linear multistep method to (2.71) with a fixed step h is (if $g \equiv 0$) equivalent to the use of a certain (ρ, σ) -reducible quadrature rule to (2.69) in the quadrature method whilst similar parallels hold for a Runge-Kutta method applied to (2.71) and the extended R-K method applied to (2.69). *This equivalence indicates the relevance of the stability analysis for methods for o.d.e.s (including recent concepts such as q -stability and algebraic stability, [6, pp 95-111]).*

The benefit in Bownds' technique [10], [11] arises from the ability to use automatic routines for (2.7') which may have variable order and variable step. The choice of approximation to $H(x,y,f)$ is a critical aspect of Bownds' approach.

The method proposed by Bownds can be specialized in order to treat convolution equations, in which $H(x,y,f) = k(x-y)H^*(y,f(y))$, this kernel being approximated by one which is implicitly of the form $\sum \lambda_r(x-y)^r H^*(y,f(y))$, via an approximation to $k(z)$ in terms of a finite weighted sum of shifted Chebyshev polynomials. (Golberg [14] shows how the Volterra equation of the second kind can be reduced to a system of differential equations when $H(x,y,f) = \sum \alpha_r k_r(x-y)H^*(y,f(y))$ provided that the functions $k_r(z)$ satisfy a system of differential equations

$$k'_r(z) = \sum a_{rs} k_s(z),$$

so that the use of Chebyshev polynomials in Bownds' method is not an essential feature of the technique.)

We mention Bownds' techniques because it indicates a connection between general Volterra problems and ordinary differential equations which is exploited in certain approaches to the analysis of numerical stability [Wolkenfelt 34].

2.6 γ -Modified Methods.

In this section we shall consider the γ -modified mixed-quadrature - Runge-Kutta methods proposed by van der Houwen [19] (see also [18]) with the aim of improving stability behaviour. The motivation for the modified mixed methods arose from the observation that frequently A-stable R-K tableau employed in mixed quadrature R-K methods gave rise to methods which were not A-stable when applied to the integrated forms of a differential equation (2.41). (For the present, A-stability in the sense used by Lambert [23] suffices.) Thus, the purpose of deriving the modified mixed methods is to restore A-stability when the R-K tableau is A-stable. A slight loss of order of accuracy is the penalty for the increased stability.

We recall the discretization of (2.45), with $x \in \{\tau_j\}$, to produce formulae of the type (2.52) in which the weights $\Omega_{jk}[Q, A]$ of (2.54) are employed. When $r = p$, $\tau_j = ih + \theta_p h = (i+1)h$ and $\Omega_{(i+1)(p+1),k}$ are given in terms of $\{\omega_{ik}\}$ and $A_{p,s}$, $s = 0, 1, \dots, p-1$.

From (2.45), we obtain

$$\tilde{f}_j = h \sum_{k=0}^{(i+1)(p+1)} \Omega_{jk} H(\tau_j, \tau_k, \tilde{f}_k) + g(\tau_j), \quad j = 1, 2, 3, \dots \quad (2.81)$$

Replacing i by $i-1$ and taking $r = p$

$$\tilde{f}_{i(p+1)} = h \sum_{k \geq 0} \Omega_{i(p+1),k} H(ih, \tau_k, \tilde{f}_k) + g(ih) \quad (2.82)$$

Then the modified methods are defined by the formulae (obtained by subtracting $\gamma_r \times$ equation (2.82) from equation (2.81))

$$\tilde{f}_j^{[\gamma]} = g(\tau_j) + h \sum_{k \geq 0} \Omega_{j,k} H(\tau_j, \tau_k, \tilde{f}_k^{[\gamma]}) + \gamma_r c_i^{[\gamma]} \quad (2.83)$$

$$c_i^{[\gamma]} = \tilde{f}_{i(p+1)}^{[\gamma]} - h \sum_{k=0}^{i(p+1)} \Omega_{i(p+1),k} H(ih, \tau_k, \tilde{f}_k^{[\gamma]}) - g(ih) \quad (2.84)$$

in which a parameterizing vector is selected such that

$$\gamma = [\gamma_0, \gamma_1, \dots, \gamma_p]^T, \quad 0 \leq \gamma_r \leq 1. \quad (2.85)$$

We note that $c_i^{[\gamma]}$ provides an 'estimate' of the accuracy of the method at the end of the i -th step and is based upon previous computed values $\tilde{r}_k^{[\gamma]}$, $k = 0, 1, \dots, i(p+1)$. Thus, from (2.83), we may regard the estimate at the end of the i -th step as influencing the $(i+1)$ -th step.

If we consider the choice $H(x, y, v) = \lambda v$ in (2.83) with $\gamma = [1, 1, \dots, 1]^T$ it follows that

$$\tilde{r}_j^{[\gamma]} = \tilde{r}_{i(p+1)}^{[\gamma]} + \lambda h \sum_{k=i(p+1)+1}^{(i+1)(p+1)} \Omega_{jk} \tilde{r}_k^{[\gamma]}. \quad (2.86)$$

We see that the γ -modified method with $\gamma_r \equiv 1$ ($r = 0, 1, \dots, p$) reduces, when applied to the equation

$$f(x) = g(x) + \lambda \int_0^x f(y) dy,$$

to the extended R-K method for this equation. In particular, we observe the A-stability of the modified method with $\gamma_r \equiv 1$ when the R-K tableau is A-stable.

For further discussion of stability considerations which may motivate the γ -modified Runge-Kutta methods and related developments we may refer to van der Houwen, Wolkenfelt & Baker [21], [39], Wolkenfelt [36] and van der Houwen & te Riele [20]. Some insight on the convergence of the methods may also be found in van der Houwen [19] and van der Houwen, Wolkenfelt & Baker [21].

REFERENCES

1. BAKER, C.T.H. Initial value problems for Volterra integro-differential equations. pp.296-307 in [16]. HALL, G. & WATT, J.M. (eds.)
2. BAKER, C.T.H. The numerical treatment of integral equations. Clarendon Press, Oxford (1977, reprinted 1978).
3. BAKER, C.T.H. & KEECH, M.S. Stability regions in the numerical treatment of Volterra integral equations. Siam J. Numer. Anal. Vol. 15, No.2 (1978).
4. BAKER, C.T.H., MAKROGLOU, A., & SHORT, E. Regions of stability in the numerical treatment of Volterra integro-differential equations. Siam J. Numer. Anal. Vol. 16, No.6 (1979).
5. BAKER, C.T.H. & MILLER, G.F. (eds.) Treatment of Integral Equations by Numerical Methods. Academic Press, London (1982).
6. BAKER, C.T.H. & PHILLIPS, C. (eds.) The numerical solution of non-linear problems. Clarendon Press, Oxford (1981).
7. BAKER, C.T.H. & WILKINSON, J.C. Stability analysis of Runge-Kutta methods applied to a linear Volterra integral equation. J. Austral. Math. Soc. (B) 22, pp.515-538 (1981).
8. BELTYUKOV, B.A. An analogue of the Runge-Kutta methods for the solution of a non-linear integral equation of the Volterra type. Differential Equations 1 pp.417-433 (1965).
9. BOWNDS, J.M. On an initial-value method for quickly solving Volterra Integral Equations: A Review. J. Opt. Theory Appl. 24, No.1, pp.133-151 (1978).
10. BOWNDS, J.M. Theory and performances of a subroutine for solving Volterra integral equations. Math. Rept., Univ. of Arizona, Tucson (1981).
11. BOWNDS, J.M. Comments on the performance of a Fortran sub-routine for certain Volterra Equations. pp.163-167 in [5]. BAKER, C.T.H. & MILLER, G.F. (eds.)
12. BRUNNER, H., HAIRER, E. & NØRSETT, S.P. Runge Kutta theory for Volterra integral equations of the second kind. Math. Comp. 39 pp. 147-163 (1982).
13. BRUNNER, H. & LAMBERT, J.D. Stability of numerical methods for Volterra integro-differential equations. Computing 12 pp.75-89 (1974).
14. GOLBERG, M.A. On a method of Bownds' for solving Volterra integral equations. J. Opt. Theory Appl. 24, No.1, pp.245-256 (1978).
15. HAIRER, E. Extended Volterra - Runge-Kutta Methods. pp.221-231 in [5]. BAKER, C.T.H. & MILLER, G.F. (eds.).

16. HALL, G. & WATT, J.M. (eds.). Modern numerical methods for ordinary differential equations. Clarendon Press, Oxford (1976).
17. de HOOG, F. & WEISS, R. On the solution of Volterra integral equations of the first kind. Numer. Math. 21, pp.22-32 (1973).
18. van der HOUWEN, P.J. On the numerical solution of Volterra integral equations of the second kind. Rep. NW42/77, Math. Centrum, Amsterdam (1977).
19. van der HOUWEN, P.J. Convergence and stability analysis of Runge-Kutta type methods for Volterra integral equations of the second kind. Rep. NW 83/80, Math. Centrum, Amsterdam (1980) and BIT 20 pp.375-377.
20. van der HOUWEN, P.J. & te RIELE, H.J.J. Linear Multistep Methods for Volterra integral equations of the second kind. pp.79-93 in [5]. BAKER, C.T.H. & MILLER, G.F. (eds.).
21. van der HOUWEN, P.J., WOLKENFELT, P.H.M., & BAKER, C.T.H. Convergence and stability analysis for modified Runge-Kutta methods in the numerical treatment of second kind Volterra integral equations. IMA J. Numer. Anal. 1 pp.303-328 (1981).
22. KOBAYASI, M. On numerical solution of the Volterra integral equations of the second kind by linear multistep methods. Rep. Statis. Applic. Res. (JUSE) 13 pp.1-21 (1966).
23. LAMBERT, J.D. Computational methods in ordinary differential equations. John Wiley, London (1973).
24. LINZ, P. Linear multistep methods for Volterra integro-differential equations. J. Ass. Comput. Mach. 16, pp.295-301 (1969).
25. LUBICH, Ch. On the stability of linear multi-step methods for Volterra integral equations of the second kind. pp.233-238 in [5]. Baker, C.T.H. & MILLER, G.F. (eds.).
26. MATTYS, J. A-stable linear multistep methods for Volterra integro-differential equations. Numer. Math. 27 pp.85-94 (1976).
27. NEVANLINNA, O. On the numerical solution of some Volterra equations on infinite intervals. Rep. 2. Institut Mittag-Leffler, Djursholm (1976).
28. NEVANLINNA, O. Positive quadratures for Volterra equations. Computing 16 pp.349-357 (1976).
29. NEVANLINNA, O. On the stability of discrete Volterra equations. pp.139-147 in [5]. BAKER, C.T.H. & MILLER, G.F. (eds.).

30. POUZET, P. Méthode d'intégration numérique des équations intégrales et intégrro-différentielles du type de Volterra de seconde espèce. Formules de Runge-Kutta In Symposium on the numerical treatment of ordinary differential equations, integral and integro-differential equations pp.362-368, Birkhauser-Verlag, Basel (1960).
31. STETTER, H. Analysis of discretization methods for ordinary differential equations. Springer, New York (1973).
32. TAVERNINI, L. One step methods for the numerical solution of Volterra functional differential equations. Siam J. Numer. Anal. 8, pp.786-795 (1971).
33. WOLKENFELT, P.H.M. Linear multistep methods and the construction of quadrature formulae for Volterra integral and integro-differential equations. Rep. NW76/79, Math. Centrum, Amsterdam (1979).
34. WOLKENFELT, P.H.M. The numerical analysis of reducible quadrature methods for Volterra integral and integro-differential equations. Dissertation, Math. Centrum, Amsterdam (1981).
35. WOLKENFELT, P.H.M. On the relation between the repetition factor and numerical stability of direct quadrature methods for second kind Volterra integral equations. Rep. NW17/81, Math. Centrum, Amsterdam (1981). See also [34].
36. WOLKENFELT, P.H.M. Modified multilag methods for Volterra functional equations. Rep. NW 108/81, Math. Centrum, Amsterdam (1981). See also [34].
37. WOLKENFELT, P.H.M. Reducible quadrature methods for Volterra integral equations of the first kind. BIT 21, pp.232-241 (1981). See also [34].
38. WOLKENFELT, P.H.M. The construction of reducible quadrature rules for Volterra integral and integro-differential equations, IMA Journal of Num. Anal. (1982). See also [34].
39. WOLKENFELT, P.H.M., van der HOUWEN, P.J. & BAKER, C.T.H. Analysis of numerical methods for second kind Volterra equations by imbedding techniques. J. Int. Eqns. 3 pp.61-82 (1981).

CHAPTER 3

3. Basic stability analysis of Runge-Kutta methods for Volterra integral equations of the second kind.

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3. Basic stability analysis of Runge-Kutta methods for Volterra integral equations of the second kind.

3.1 Introduction

Our main purpose in this chapter is to reveal the application and analysis of certain stability definitions applied to Runge-Kutta methods for a second kind Volterra integral equation of a simple form, now generally described in the literature as the "basic test equation". Thus we have two prerequisites: To discuss the class of 'test' equations which might be used and to introduce stability definitions and examine their application when applied to a certain class of methods.

In §3.2 we have some preliminary remarks on stability with an attempt to place on a firm foundation our choice of 'basic test equation'. The results of our investigations form the basis of this work although there have been some subsequent developments [18] which will also be briefly discussed. Nevertheless, we shall seek to confirm that the analysis of our chosen 'basic test equation' provides a necessary foundation on which to develop a theory for more involved test equations where the complexity of the analysis can obscure the insight. The results for our basic test equation not only lead to genuine understanding but have stimulated the production of new methods.

In §3.3 we summarise some important results in the study of recurrence relations and elaborate on some relevant remarks on the structure of quadrature weights.

In §3.4 we introduce some of the stability definitions which we intend using in subsequent sections. The relevance of these definitions is examined in connection with Runge-Kutta methods for the second kind Volterra integral equations. Some of our results obtained for our test equation mirror results already well known when Runge-Kutta

methods are used for the numerical solution of initial value problems of ordinary differential equations.

In §3.5 and §3.6 we introduce an analysis of the mixed Runge-Kutta methods with reducible and block-reducible quadrature rules. We find that the structure of the methods used, results in recurrence relations and stability polynomials which permit certain stability results to be stated for our 'basic test equation'.

3.2 Preliminary Remarks on Stability

The aim of this section is to motivate discussions in subsequent sections in this chapter.

We consider the Volterra integral equation of the second kind

$$f(x) = g(x) + \int_0^x H(x,y, f(y))dy, \quad (x \geq 0) \quad (3.1)$$

When solving such an equation numerically we wish to know whether the behaviour of the discretized equation, as $x \rightarrow \infty$, reflects that of the original integral equation (3.1). For example, will the numerical solution converge to zero or remain bounded if the exact solution does? The study of stability can be related to the qualitative behaviour of solutions.

In our discussions of stability, consideration will be restricted to the linear version of (3.1)

$$f(x) = g(x) + \lambda \int_0^x K(x,y) f(y)dy, \quad (x \geq 0) \quad (3.2)$$

The earliest stability studies in the numerical treatment of such equations ([4], [7]) have been concerned with the performance of methods applied to an equation which has become generally known in the literature as the 'basic test equation'

$$f(x) - \lambda \int_0^x f(y) dy = g(x). \quad (3.3)$$

(It reduces to $f'(x) = \lambda f(x) + g'(x)$, $f(0) = g(0)$.) This test equation is open to some objections because of its simplicity and the fact that certain typical features of Volterra integral equations are not present. We shall see that attempts have been made to search for other more realistic problems. Nevertheless, we shall argue that stability conditions for methods applied to (3.3) provide initial criteria against which the versatility of methods may be judged. Thus methods exhibiting poor stability properties on (3.3) are unlikely to do well on more difficult problems. Further, the stability study of methods for (3.3) provides insight into a structure which is valuable for the study of more general equations ([6], [29]).

3.2.1 Stability of the continuous problem

The study of stability of the solution of the integral equation (3.1) is concerned with the sensitivity of $f(x)$ to perturbations in the problem. For an introduction to this topic we refer to Tsalyuk [26]. Particular attention has been given to the effect on $f(x)$ of a perturbation $\delta g(x)$ in $g(x)$ as $x \rightarrow \infty$ and various stability definitions have been given arising from the restrictions on the class of perturbations $\delta g(x)$ to various normed linear spaces (See, for example, Bownds and Cushing [9].)

For the linear version (3.2), progress in the stability analysis can be made by analysing the resolvent or differential resolvent. We recall from §1.9.5 the familiar results

$$f(x) = g(x) + \lambda \int_0^x R_\lambda(x, y) g(y) dy \quad (3.4)$$

$$\text{where } R_\lambda(x,y) = \int_0^x K(x,z) R_\lambda(z,y) dz + K(x,y) \quad (3.5)$$

(The resolvent kernel may be developed from the Neumann series:

$$R_\lambda(x,y) = K(x,y) + \lambda K^2(x,y) + \lambda K^3(x,y) \dots \text{ where}$$

$$K^r(x,y) = \int_0^x K(x,z) K^{r-1}(z,y) dz$$

In practice, it is difficult to derive properties of the resolvent kernel for an arbitrary kernel $K(x,y)$.

Even assuming the linearity of K it cannot be expected that there will be a general solution to the problem of stability. Hence, one tries to investigate stability problems for special classes of equations where one hopes to gain insight under simplifying assumptions on the nature of the kernel.

Several papers have been published in which the essence of the approach turns out to be analysis of a numerical method for ordinary differential equations to which the integral equation is more or less equivalent when the same numerical method is used and certain types of kernel functions are involved. Thus, Baker and Keech [7] and Van der Houwen [14], respectively, have produced stability results for particular classes of methods and kernel functions varying from simple linear functions such as $H(x,y,f) = af$ and $H(x,y,f) = (ax+b)f$, a and b constant to rather more general separable kernels of the form

$$H(x,y,f) = \sum_{i=1}^r X_i(x) Y_i(y,f), \quad (3.6)$$

(cf. Baker [6] and Van der Houwen & Wolkenfelt [15]). Particular examples of kernels of the form (3.6) such as the polynomial convolution kernel

$$K(x,y) = K(x-y) = \sum_{r=0}^R \lambda_r (x-y)^r \quad (3.7)$$

and

$$k(x,y) = \lambda_0 + \lambda_1^0(x) + \lambda_1^1(y) \quad (3.8)$$

have also been studied in Amini and Baker [1] and Amini, Baker, Van der Houwen & Wolkenfelt [2]. The assumption of the form of polynomial convolution kernel (3.7) in (3.1) produces equations

$$f(x) = g(x) + \int_0^x \sum_{r=0}^R \lambda_r(x-y)^r f(y) dy \quad (3.9)$$

which permit a reduction of (3.1) to a system of ordinary differential equations and the analogue of stability results for differential equations can be established for (3.1). Results for (3.3) can also be deduced from those of (3.9).

3.2.2 Stability of the numerical method

If, in (3.2), $g(x)$ is perturbed by a small amount $\delta g(x)$, then the resulting change $\delta f(x)$ in $f(x)$ may be obtained from (3.4) by

$$\delta f(x) = \delta g(x) + \lambda \int_0^x R_\lambda(x,y) \delta g(y) dy. \quad (3.10)$$

When direct quadrature methods are applied to (3.2) we obtain the quadrature equations (see (2.4))

$$\tilde{f}_i = g(ih) + h \sum_{k=0}^i \omega_{ij} K(ih, kh) \tilde{f}_k, \quad i = 1, 2, 3, \dots \quad (3.11)$$

and the stability analysis involves the study of the effect of perturbations δg_n in $g(nh)$ ($n = 0, 1, \dots$) on the resulting changes δf_n in \tilde{f}_n when

$$\delta f_n = \delta g_n + \lambda h \sum_{j=0}^n \omega_{nj} K(nh, jh) \delta f_j \quad (3.12)$$

The perturbation δf_n depends on all previous perturbations δf_j , $j = 0(1)n-1$. Miller [19] has given a theory of finite recurrence equations, in which a perturbation δf_n depends on a finite number of perturbations $\delta f_{n-1}, \dots, \delta f_{n-k}$ and this theory may

provide insight into the more general problem of (3.12) when the kernel and quadrature weights satisfy suitable conditions.

Two approaches to the stability analysis may be found in the literature. These are described in the following sections.

3.2.3 Numerical stability for small h

Numerical stability "for small h ", in the sense of Linz [17] and Noble [22] requires the perturbation sensitivity of the numerical method to be "roughly equivalent" to the perturbation sensitivity of the original problem. The approach of Linz and Noble is based on the asymptotic expansion of the global discretization error. Before pursuing this direction it may be helpful to state here their respective stability definitions.

Definition 3.1 (Linz [17])

A step-by-step method for (3.1) is numerically stable if the error growth is roughly equivalent to that of the solution of the variational equation of (3.1). If there exist some equations for which the error grows much faster than the solution of the variational equation of (3.1) then the method must be considered numerically unstable.

Definition 3.2 (Noble [22])

A step-by-step method for solving a Volterra integral equation is said to be unstable if the error in the computed solution has dominant spurious components introduced by the numerical scheme. The meaning of the term "spurious" will be remarked upon later.

Now we turn our attention to the analysis of the global discretization error. Considering a (ρ, σ) - reducible quadrature method of order p (defined in §2.2.1) the asymptotic expansion of the

global discretization error $e(x_n) = \tilde{f}_n - f(x_n)$ may be shown to have the form

$$e(x_n) = h^p \sum_{i=1}^s \mu_i^n e_p^{(i)}(x) + O(h^{p+1}), \quad (3.13)$$

where $\mu_1 = 1$, μ_2, \dots, μ_s are the essential zeros of ρ , that is, are of modulus 1. $e_p^{(i)}(x)$ satisfies

$$e_p^{(i)}(x) = g_p^{(i)}(x) + \gamma_i \int_0^x K(x, y) e_p^{(i)}(y) dy \quad (3.14)$$

and $K(x, y) = \frac{\partial}{\partial f} H(x, y, f(y))$.

The quantities γ_i are the growth parameters (Henrici [12, p237]) defined as

$$\gamma_i = \sigma(\mu_i) / \mu_i \rho'(\mu_i), \quad i = 1, 2, \dots, s$$

The functions $g_p^{(i)}(x)$ are related to the errors in the starting values and to local quadrature errors.

Note that if in (3.1) $g(x)$ is perturbed by a small amount $\delta g(x)$ then $f(x)$ changes by a small amount $\delta f(x)$ where, with $K(x, y) = \frac{\partial}{\partial v} H(x, y, v) \Big|_{v=f(y)}$

$$\delta f(x) \approx \delta g(x) + \int_0^x K(x, y) \delta f(y) dy$$

which is of the form (3.2) with $\lambda=1$.

Accordingly, we have (3.10) with $\lambda=1$.

The component $e_p^{(1)}(x)$ associated with $\gamma_1 = 1$ is the principal error component which mimics the effect of an analytical perturbation. The remaining components $e_p^{(2)}(x), \dots, e_p^{(s)}(x)$ associated with $\gamma_2, \dots, \gamma_s$ are the "spurious" error components. Clearly, if $|e_p^{(i)}(x)| \gg |e_p^{(1)}(x)|$ for some i ($2 \leq i \leq s$) then, in the sense of Linz and Noble, the method is numerically unstable.

The advantage of the approach based on the asymptotic expansion of the global discretization error is that it is applicable to general equations (3.2) without additional constraints on the kernel K and the function g ; the values of the growth parameters are clearly crucial for numerical stability. To reinforce this latter point we consider the basic test equation

$$f(x) = g(x) + \lambda \int_0^x f(y) dy. \quad (3.15)$$

whose solution is given by

$$f(x) = g(x) + \lambda \int_0^x \exp(\lambda(x-y)) g(y) dy \quad (3.16)$$

Suppose there is a value $\gamma_i \neq 1$ for some $i \neq 1$. Then, from (3.14) and (3.16) there is an associated spurious error component $e_p^{(i)}(x)$ given by

$$e_p^{(i)}(x) = g_p^{(i)}(x) + \gamma_i \lambda \int_0^x \exp(\gamma_i \lambda(x-y)) g_p^{(i)}(y) dy. \quad (3.17)$$

The problem (3.15), whose solution is (3.16), is well-conditioned with respect to bounded perturbations of g if $\text{Re } \lambda < 0$. However, we observe from (3.17) that if $\gamma_i \neq 1$ ($i \neq 1$) and $\text{Re}(\gamma_i \lambda) > 0$ then the global error has a spurious component $e_p^{(i)}(x)$ which is exponentially increasing. Furthermore, in his thesis Linz [17] conjectured that quadrature methods using rules with repetition factor one (see Definition 2.4(a)) tend to be numerically stable, whilst those with repetition factor greater than one tend to be numerically unstable. This was a useful premise to stimulate further analysis and Wolkenfelt [28] has investigated this conjecture. First of all he shows that the natural interpretation of Linz's conjecture

is too strong by providing, in the following theorem, a more quantitative definition of the numerical stability concept.

Theorem 3.1 Wolkenfelt [28, p.114]

A reducible quadrature method of the form (3.11) is numerically stable (for small h) (in the sense of Linz and Noble) if each essential zero of ρ has a growth parameter equal to one; the method is weakly stable (for small h) (or numerically unstable in the terminology of Linz and Noble) if there exists at least one essential zero of ρ whose growth parameter is different from one.

Secondly, Wolkenfelt has demonstrated that methods with an asymptotic repetition factor of one are always numerically stable in the sense of Linz and Noble but that methods with an exact or asymptotic factor greater than one can also be numerically stable.

Example 3.1(a)

An example of a numerically stable method which has an exact repetition factor of two is obtained by taking $\rho(\mu) = \mu^2 - 1$ and $\sigma(\mu) = \mu^2 + 1$. (The growth parameters associated with the essential zeros $\mu_1 = 1$ and $\mu_2 = -1$ are both equal to one.)

Example 3.1 (b) (Wolkenfelt [28])

Consider the (ρ, σ) - reducible quadrature method with $\rho(\mu) = (\mu^2 - 1)(\mu - \frac{1}{3})$ and $\sigma(\mu) = (\mu^2 - 2/3\mu + 1)$. The weights have an asymptotic repetition factor of 2 (since the essential zeros of ρ satisfy $\mu^2 = 1$) and the method is numerically stable as the growth parameters associated with the essential zeros of ρ are both equal to one.

However, the disadvantage of this stability analysis is its asymptotic nature (as $h \rightarrow 0$) and therefore the conclusions drawn may not hold for particular values of h . We need to gain greater insight into an appropriate choice of h by considering regions of

stability and a class of test equations.

3.2.4 Numerical stability for fixed h

We have already indicated that in order to obtain insight into the general stability problem (3.12), special cases of (3.2) have been considered in the literature. In §3.2 and §3.2.1 we cited some of the special cases of $K(x,y)$ which may be considered, the most simple case arising from the choice of $K(x,y) = 1$, $\lambda \in \mathbb{C}$ yielding the 'basic test equation' (cf. Baker and Keech [7])

$$f(x) = g(x) + \lambda \int_0^x f(y) dy. \quad (3.18)$$

However, we cannot assume that a method which is suitable for our rather special test equation (3.3) is suitable for more complicated equations. To this end attempts have been made to provide a firmer foundation to the stability analysis based on (3.3) by considering more general kernel functions. Earlier we remarked that Van der Houwen [14] considered kernel functions of the form

$$H(x,y,f) = (a+bx)f, \quad a \text{ and } b \text{ constant.} \quad (3.19)$$

Baker [6] and Van der Houwen & Wolkenfelt [15] considered the class of finitely decomposable kernels of the form

$$H(x,y,f) = \sum_{i=0}^r X_i(x) Y_i(y,f). \quad (3.20)$$

The relevance of (3.19) may also be disputed as the equation can be reduced to a second order differential equation. However the form of stability analysis for (3.19) provided understanding when dealing with kernel functions of the form (3.20). In [15] Van der Houwen and Wolkenfelt compare the stability conditions derived for kernels of the form $H = \lambda f$ with those derived for (3.20) and conclude that the first class of kernel functions gives a rough indication of the

stability behaviour of the numerical scheme. Hence (3.3) may be considered as a first sieve for the selection of an appropriate scheme for the solution of Volterra integral equations of the second kind.

Other contributions of the stability theory for fixed h exist, which vary in the class of methods considered and in the special case of (3.2) adopted as a test equation. For a brief survey of some of these approaches we may refer to Amini, Baker, Van der Houwen and Wolkenfelt [2]. Since our earlier investigations, but prior to recording them here, there have been several new approaches to stability analysis which we shall discuss briefly in the following section.

3.2.5 Further contributions to the stability theory for fixed h .

Lubich [18] has observed that, in applications, one often encounters convolution equations of the form

$$f(x) = g(x) + \int_0^x k(x-y)\varphi(y, f(y))dy, \quad x \geq 0 \quad (3.21)$$

In this subsequent stability analysis he considers the linear case where $\varphi(y, f(y)) = f(y)$ and makes use of the Paley and Wiener theorem (Theorem 1.10) which provides a result on the asymptotic stability of Volterra integral equations of convolution type. The theorem is repeated here for convenience.

Theorem 3.2 (Paley and Wiener [24])

Consider the equation

$$f(x) - \int_0^x k(x-y) f(y)dy = g(x), \quad x \geq 0 \quad (3.22)$$

where the kernel $k(x)$ belongs to $L^1(0, \infty)$. Then we have

$$f(x) \rightarrow 0 \quad \text{whenever} \quad g(x) \rightarrow 0 \quad (x \rightarrow \infty)$$

if and only if

$$\int_0^{\infty} e^{-wx} k(x) dx \neq 1 \quad \text{for } \operatorname{Re} w \geq 0. \quad (3.23)$$

Lubich investigated the asymptotic behaviour of the numerical solutions obtained by applying linear multi step methods with a fixed step size h to the following class of Volterra integral equation

$$f(x) = g(x) + \lambda \int_0^x k(x-y)f(y)dy \quad (x \geq 0) \quad (3.24)$$

where $\operatorname{Re} \lambda < 0$ and the continuous kernel $k(x) \in L^1(0, \infty)$ with $\{k(nh)\}_{n=1}^{\infty} \in \mathcal{Q}'$ is positive definite. That is, for all $n \geq 1$,

$\sum_{j,m=1}^n k(\xi_j - \xi_m) z_j \bar{z}_m \geq 0$ for any choice of real numbers ξ_1, \dots, ξ_n and complex numbers z_1, \dots, z_n : $k(x)$ is extended to the negative real axis by $k(x) = \overline{k(-x)}$. By Bochner's theorem this is equivalent to (3.23). Examples of positive definite functions are convex non-negative, non increasing functions on $[0, \infty)$, $\cos x$ and $\exp(-x^2)$. By the use of the Paley and Wiener theorem, it can be demonstrated that positive definite kernels form "the largest class of linear convolution kernels" such that

$f(x) \rightarrow 0$ (is bounded) whenever $g(x) \rightarrow 0$ (is bounded) for all λ with $\operatorname{Re} \lambda < 0$.

For initial value problems in ordinary differential equations, a method is called A-stable if the left half-plane $\operatorname{Re} z < 0$ is contained in the stability region for the basic equation. Lubich has shown that, in the application of an A-stable linear multistep method to an integral equation with positive definite kernel, the stability region of the linear multistep method is not preserved. However, in general, $A(\alpha)$ -stability (that is, where the stability region contains the sector $\arg z \in (\pi - \alpha, \pi + \alpha)$ with $\alpha \in (0, \pi/2)$) does carry over to a restricted class of integral equations (3.24) where the

continuous L' kernel $k(x)$ is completely monotonic. That is,

$(-1)^s k^s(x) \geq 0$ for $s = 0, 1, 2, \dots$ and $x \geq 0$, where

$k^s(x)$ represents the s th derivative of $k(x)$.

In addition to Lubich's work Nevanlinna [20], [21] has also considered test equations with non-separable kernels. However, Nevanlinna's papers and that of Lubich deal only with multistep methods. In [11] Hairer and Lubich show to what extent the results of [18] can be adapted for extended Runge-Kutta methods when applied to selected convolution equations with positive definite L' kernels.

3.2.6 Concluding remarks

We remain convinced that the development of the stability analysis for (3.3) is useful and relevant. We accept that there are limitations to the use of (3.3) as a test equation and these have been cited in Baker [4] and repeated elsewhere. To overcome some of these limitations more general test equations can be, and have been, mentioned in earlier sub-sections in this chapter. Nevertheless, the stability analysis of the simple test equation gives genuine insight into the structure and complexity of the analysis arising from more general test equations. Indeed, it now seems to be generally accepted that suitability of a method for the test equation (3.3) is a necessary (but not sufficient) requirement for an all purpose method.

3.3 Further Remarks on Structure

In this section we refer to two prerequisites which will be useful in subsequent sections. The first refers to the type of recurrence relations which arise; the second deals with the special structure of the quadrature weights which will also be exploited.

3.3.1 Recurrence Relations

When determining stability regions of numerical methods for initial value problems in ordinary differential equations one considers the

effect of applying the numerical scheme to the test equation:

frequently $f'(x) = \lambda f(x)$, λ real or complex. Thus Lambert [16]

illustrates that a (ρ, σ) linear multistep method associated with the polynomials $\rho(z) = \sum_{i=0}^k \alpha_i \mu^{k-i}$ and $\sigma(z) = \sum_{i=0}^k \beta_i \mu^{k-i}$ for this particular test equation gives the formulae

$$\sum_{i=0}^k (\alpha_i - \lambda h \beta_i) \tilde{f}_{n+i} = 0, \quad n = 0, 1, 2, \dots \quad (3.25)$$

where we seek $\tilde{f}_r \approx f(rh)$, given suitable starting values $\tilde{f}_0, \dots, \tilde{f}_{k-1}$.

If $\alpha_k - \lambda h \beta_k \neq 0$ and $\gamma_i = (\alpha_i - \lambda h \beta_i) / (\alpha_k - \lambda h \beta_k)$ then (3.25) may be written in the form (cf. Baker & Keech [7])

$$\begin{bmatrix} \tilde{f}_{n+k} \\ . \\ . \\ . \\ \tilde{f}_{n+1} \end{bmatrix} = \begin{bmatrix} \gamma_{k-1} & \gamma_{k-2} & . & . & . & \gamma_0 \\ 1 & 0 & . & . & . & 0 \\ 0 & 1 & & & & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & . & . & . & 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{f}_{n+k-1} \\ . \\ . \\ . \\ \tilde{f}_n \end{bmatrix} \quad (3.26)$$

These recurrence relations may be expressed as

$$\tilde{\varphi}_{k+1} = \tilde{M} \tilde{\varphi}_k \quad \text{where} \quad \tilde{\varphi}_k = [\tilde{f}_{n+k-1}, \dots, \tilde{f}_n]^T \quad (3.27)$$

and \tilde{M} is a fixed square matrix, independent of k . The usual stability criterion in terms of the zeros of $\rho(z) - \lambda h \sigma(z)$ imposes a requirement that the zeros of $\{z^k - \sum_{i=0}^{k-1} \gamma_i z^i\}$ (equivalently, the eigenvalues of \tilde{M}) lie in/on the unit disc. We return to this point later.

Here, we are concerned to illustrate how we shall gain insight into relevant aspects of error propagation when developing the stability analysis of numerical schemes applied to integral equations.

We recall that, when extended or mixed R-K methods are used for the discretization of Volterra Integral Equations of the second kind, formulae of type (2.52) are produced where the 'weights' are denoted by (2.53) or (2.54) respectively. In addition, the quadrature methods of (2.4) fit into the framework (2.52) where the weights are given by (2.55). For convenience we reproduce the formulae of type (2.52) here:

$$\tilde{f}_j = h \sum_{k \geq 0} \Omega_{jk} H(\tau_j, \tau_k, \tilde{f}_k) + g(\tau_j), \quad j = 1, 2, 3, \dots \quad (3.28)$$

Due to the simple form of our 'basic test equation' (3.3), when differencing procedures are applied to (3.28), under certain conditions such as the structure in the weights Ω_{jk} , recurrence relations will be yielded of the form

$$\underset{\sim}{\Phi}_{k+1} = \underset{\sim}{M} \underset{\sim}{\Phi}_k + \underset{\sim}{\gamma}_k, \quad k = 0, 1, 2, \dots \quad (3.29)$$

where $\underset{\sim}{M} \equiv \underset{\sim}{M}(\lambda h)$ and $\underset{\sim}{\Phi}_0 = \underset{\sim}{\gamma}$ is given. The components of the vectors $\underset{\sim}{\Phi}_k$ will be successive values of \tilde{f}_j the approximations to $f(jh)$. Stability definitions given in later sections are introduced in order to permit the study of relationship between the solution of (3.29) and the solution of a perturbed recurrence

$$\hat{\underset{\sim}{\Phi}}_{k+1} = \underset{\sim}{M} \hat{\underset{\sim}{\Phi}}_k + \underset{\sim}{\gamma}_k + \underset{\sim}{\delta}_k \quad (3.30)$$

(where the term $\underset{\sim}{\delta}_k$ stems from the introduction of rounding error or truncation error). Clearly, if $\underset{\sim}{\xi}_k = \hat{\underset{\sim}{\Phi}}_k - \underset{\sim}{\Phi}_k$ we have $\underset{\sim}{\xi}_{k+1} = \underset{\sim}{M} \underset{\sim}{\xi}_k + \underset{\sim}{\delta}_k$, so that the study of error propagation (the behaviour of the sequence $(\underset{\sim}{\xi}_k)$ in response to $(\underset{\sim}{\delta}_k)$) is in effect the study of the behaviour of the solution of equations of the form (3.29). For further elaboration we refer to Baker [4].

3.3.2 Partitioning - Further Remarks on the structure of weights

Here we make some additional observations on the weights Ω_{jk} (A)

defined in (2.53) for the classical extended R-K formulae and of the weights $\Omega_{jk} [Q, \tilde{A}]$ defined in (2.54) for the mixed quadrature R-K method. Later, we shall find it helpful to refer to the following structures. The array of values $\Omega_{jk} (\tilde{A})$ which are defined in (2.53) as

$$\Omega_{jk} (\tilde{A}) = \begin{cases} A_{pt}, & 0 < k \leq i(p+1) \\ A_{rt}, & i(p+1) < k \leq (i+1)(p+1) \\ 0 & \text{otherwise} \end{cases} \quad (3.31)$$

where $t \equiv (k-1) \bmod (p+1)$, $r \equiv (j-1) \bmod (p+1)$, may be represented in partitioned form given below:

$$\Omega_{jk} (\tilde{A}) = \begin{array}{c} \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \end{array} \begin{array}{|c|} \hline \tilde{A} \\ \hline \end{array} \\ \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \end{array} \begin{array}{|c|c|} \hline \begin{array}{c} e a_p^T \\ \sim \sim \end{array} & \tilde{A} \\ \hline \end{array} \\ \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \end{array} \begin{array}{|c|c|c|} \hline \begin{array}{c} e a_p^T \\ \sim \sim \end{array} & \begin{array}{c} e a_p^T \\ \sim \sim \end{array} & \tilde{A} \\ \hline \end{array} \\ \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \\ \hline \end{array} \\ \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \end{array} \begin{array}{|c|c|c|c|c|c|} \hline \begin{array}{c} e a_p^T \\ \sim \sim \end{array} & \begin{array}{c} e a_p^T \\ \sim \sim \end{array} & \begin{array}{c} e a_p^T \\ \sim \sim \end{array} & \cdot & \cdot & \begin{array}{c} e a_p^T \\ \sim \sim \end{array} & \tilde{A} \\ \hline \end{array} \end{array} \quad (3.32)$$

where $\underline{A} = [A_{rs}]$ is of order $p+1$; $\underline{a}_p^T = \underline{e}_p^T \underline{A}$ (the last row of \underline{A}) where $\underline{e}_0, \underline{e}_1, \dots, \underline{e}_p$ are the successive columns of the identity matrix of order $p+1$, and $\underline{e} = \underline{e}_0 + \underline{e}_1 + \dots + \underline{e}_p$. (The matrix $\underline{e} \underline{a}_p^T$ has all rows equal to the last row of \underline{A} .)

In the mixed quadrature method, whose weights are defined in (2.54) as

$$\Omega_{jk} [Q, \underline{A}] = \begin{cases} \omega_{im}, & k = m(p+1), m \leq i \\ A_{rt}, & i(p+1) < k \leq (i+1)(p+1) \\ 0 & \text{otherwise} \end{cases} \quad (3.33)$$

r, t being defined as for (3.31), i being $[(j-1)/(p+1)]$, the partitioning assumes the form:

$$\Omega_{jk}[Q, \underline{A}] = \frac{\omega_{i0}\underline{e} \mid \omega_{i1}\underline{E}_p \mid \omega_{i2}\underline{E}_p \mid \dots \mid \omega_{ii}\underline{E}_p \mid \underline{A}}{\omega_{i+1,0}\underline{e} \mid \omega_{i+1,1}\underline{E}_p \mid \omega_{i+1,2}\underline{E}_p \mid \dots \mid \omega_{i+1,i}\underline{E}_p \mid \omega_{i+1,i+1}\underline{E}_p \mid \underline{A}} \quad (3.34)$$

where the matrix \underline{E}_p is defined as

$$\underline{E}_p = \underline{e} \underline{e}_p^T.$$

With this notation, $\underline{A}_p \equiv \underline{e} \underline{a}_p^T$ can be written $\underline{E}_p \underline{A}$. (3.35)

We observe that the ω_{ik} in (3.34) frequently have a structure which has been outlined in (2.6) as follows:

$$\Omega_{jk}(Q) = \frac{\begin{array}{c|c|c|c|c|c|c|c|c|c} \underline{w} & \underline{w}_0 & & & \underline{w}_1 & \underline{w}_2 & & & \underline{w}_p & \\ \hline \underline{w} & \underline{w}_0 & \dots & & \underline{w}_0 & \underline{w}_1 & \underline{w}_2 & \dots & \underline{w}_{p-1} & \underline{w}_p \\ \hline \underline{w} & \underline{w}_0 & \dots & & \underline{w}_0 & \underline{w}_0 & \underline{w}_1 & \dots & \underline{w}_{p-2} & \underline{w}_{p-1} & \underline{w}_p \end{array}}{\dots} \quad (3.36)$$

where \tilde{W}_p is lower triangular, the partitioning being into square submatrices of the same order q . Example 2.4 provides such an illustration. Furthermore, from §2.3.5, we recall that the structure (3.36) provides an example of a more general structure where the weights $\omega_{k\nu}$ for $\nu \geq 0$ can be partitioned as:

$$\Omega_{jk}(Q) = \begin{array}{c|c|c|c|c|c} \tilde{V}_{\ell,0} & \tilde{V}_{\ell,1} & \cdots & \tilde{V}_{\ell,\ell-1} & \tilde{V}_{\ell,\ell} & \\ \hline \tilde{V}_{\ell+1,0} & \tilde{V}_{\ell+1,1} & \cdots & \tilde{V}_{\ell+1,\ell-1} & \tilde{V}_{\ell+1,\ell} & \tilde{V}_{\ell+1,\ell+1} \end{array} \quad (3.37)$$

Here, the submatrices are all of order q and relations

$$\sum_{\ell=0}^m A_{\ell} \tilde{V}_{n+1-\ell, \nu} = \tilde{B}_{n+1-\nu} \quad (3.38a)$$

are satisfied (for all n sufficiently large) with

$$A_{\nu} = B_{\nu} = 0 \quad \text{if } \nu \notin \{0, 1, 2, \dots, m\}. \quad (3.38b)$$

The matrices A_{ℓ}, B_{ℓ} are fixed matrices of order q and at least one of the matrices A_{ℓ} is required to be non-zero. An additional and natural requirement is the condition

$$\sum_{\ell=0}^m A_{\ell} \tilde{e} = 0, \quad (3.38c)$$

where $\tilde{e} = [1, 1, \dots, 1]^T$ is the sum of the columns of the identity matrix of order q .

When the relations (3.38a,b,c) are satisfied the weights are block reducible and §2.3.5 contains illustrative examples. Further elaboration may be found in Amini, Baker & Wilkinson [3].

3.4 Extended Runge-Kutta methods and stability definitions

The earliest stability studies in the numerical treatment of Volterra equations of the second kind [4,7] have been concerned with the performance of methods of applied to the basic test equation

$$f(x) - \lambda \int_0^x f(y) dy = g(x). \quad (3.39)$$

Stability conditions for methods applied to (3.39) provide initial criteria against which the versatility of methods may be judged. Further, the stability study of methods for (3.39) provides insight into a structure which is valuable for the study of more general equations [6, 29]. We believe that for a class of problems the stability properties of methods applied to equations of the form (3.39) provide practical guidance but do not pursue here the question of the scope of this class of problems.

We shall include various stability definitions as the discussion proceeds.

3.4.1 Extended R-K Methods

Classical extended methods were defined by (2.52) which for convenience, is reproduced in (3.28). The rules (2.38) are defined by (2.53) in terms of the Runge-Kutta array (2.33) and are reproduced in (3.31) or the partitioned form (3.32). We now consider the stability of the extended methods applied to (3.39) where $H(x, y, f(y)) = \lambda f(y)$.

Equations (3.28) become

$$\tilde{f}_j = \lambda h \sum_{k=0} \Omega_{jk} \tilde{f}_k + g(\tau_j), \quad j = 1, 2, 3, \dots \quad (3.40)$$

The approximate values \tilde{f}_j obtained from the extended method define vectors $\varphi_1, \varphi_2, \varphi_3, \dots$ where

$$\varphi_{i+1} = [\tilde{f}_{i(p+1)+1}, \tilde{f}_{i(p+1)+2}, \dots, \tilde{f}_{(i+1)(p+1)}]^T \quad (i = 0, 1, 2, \dots), \quad (3.41)$$

(See also [5]).

By virtue of (3.31) with $j = i(p+1) + r + 1$ and $r = 0$ we obtain from (3.40) the equation

$$\tilde{f}_{i(p+1)+1} = \lambda h [(A_{p0} \tilde{f}_1 + \dots + A_{pp} \tilde{f}_{p+1}) + (A_{p0} \tilde{f}_{(p+1)+1} + \dots + A_{pp} \tilde{f}_{2(p+1)}) + \dots]$$

$$\begin{aligned} & \dots + (A_{p0} \tilde{f}_{(i-1)(p+1)+1} + \dots + A_{pp} \tilde{f}_{i(p+1)}) + \\ & + \lambda h [A_{00} \tilde{f}_{i(p+1)+1} + \dots + A_{op} \tilde{f}_{(i+1)(p+1)}] + g(\tau_{i(p+1)+1}) \end{aligned} \quad (3.42)$$

Using (3.32) (the partitioned form of (3.31)) together with (3.41), the equation (3.42) may be written as

$$\tilde{f}_{i(p+1)+1} = \lambda h \underset{\sim}{a}_p^T \sum_{k=1}^i \underset{\sim}{\varphi}_k + \lambda h \underset{\sim}{a}_0^T \underset{\sim}{\varphi}_{i+1} + g(\tau_{i(p+1)+1}) \quad (3.43)$$

where, we recall, $\underset{\sim}{a}_p^T = \underset{\sim}{e}_p^T A$, the last row of $\underset{\sim}{A}$.

For further values of $r = 1, 2, \dots, p$ in (3.40) the following equations are produced

$$\begin{aligned} \tilde{f}_{i(p+1)+2} &= \lambda h \underset{\sim}{a}_p^T \sum_{k=1}^i \underset{\sim}{\varphi}_k + \lambda h \underset{\sim}{a}_1^T \underset{\sim}{\varphi}_{i+1} + g(\tau_{i(p+1)+2}) \\ &\cdot \\ &\cdot \\ &\cdot \\ &\cdot \end{aligned} \quad (3.44)$$

$$\tilde{f}_{(i+1)(p+1)} = \lambda h \underset{\sim}{a}_p^T \sum_{k=1}^i \underset{\sim}{\varphi}_k + \lambda h \underset{\sim}{a}_p^T \underset{\sim}{\varphi}_{i+1} + g(\tau_{(i+1)(p+1)})$$

Combining equations (3.43) and (3.44) we obtain

$$(\underset{\sim}{I} - \lambda h \underset{\sim}{A}) \underset{\sim}{\varphi}_{i+1} - \lambda h \underset{\sim}{e}_p^T \sum_{k=1}^i \underset{\sim}{\varphi}_k = \underset{\sim}{g}_{i+1}, \quad i = 0, 1, 2, \dots \quad (3.45)$$

Here $\underset{\sim}{g}_{i+1} = [g(\tau_{i(p+1)+1}), g(\tau_{i(p+1)+2}), \dots, g(\tau_{(i+1)(p+1)})]^T$

We pause to observe that equation (3.45) forms the basis of our studies.

Recall that

$$\underset{\sim}{E}_p = \underset{\sim}{e} \underset{\sim}{e}_p^T$$

$$\text{also } \underset{\sim}{E} \underset{\sim}{A} = \underset{\sim}{e} \underset{\sim}{a}_p^T \text{ and } \underset{\sim}{e}_p^T \underset{\sim}{e} = 1.$$

Applying \underline{E}_p to (3.45) for $i = 1, 2, 3, \dots$ yields the equation

$$\underline{E}_p \underline{\varphi}_i - \lambda h \underline{e} \underline{a}_p^T \sum_{k=1}^i \underline{\varphi}_k = \underline{E}_p \underline{g}_i \quad (3.46)$$

Subtracting (3.46) from (3.45) yields

$$(\underline{I} - \lambda h \underline{A}) \underline{\varphi}_{i+1} - \underline{E}_p \underline{\varphi}_i = \underline{g}_{i+1} - \underline{E}_p \underline{g}_i \quad (3.47)$$

From (2.51), $f(\tau_{i(p+1)}) = \tilde{f}_{i(p+1)} = \tilde{f}(ih)$, $j = i(p+1)+r+1$, $r = 0, 1, \dots, p$

and we observe that $\underline{E}_p \underline{\varphi}_i$ can be written $\tilde{f}(ih) \underline{e}$. The matrix

$\underline{I} - \lambda h \underline{A}$ is invertible if we exclude at most $p+1$ exceptional values of λh for which the method fails; with this exclusion we find

$$\underline{\varphi}_{i+1} = \underline{N}(\lambda h) \underline{\varphi}_i + \underline{v}_{i+1} \quad (3.48)$$

where

$$\underline{N}(\lambda h) = \underline{d}(\lambda h) \underline{e}_p^T \quad (3.49)$$

with

$$\underline{d}(\lambda h) = (\underline{I} - \lambda h \underline{A})^{-1} \underline{e} \quad (3.50)$$

and

$$\underline{v}_{i+1} = (\underline{I} - \lambda h \underline{A})^{-1} (\underline{g}_{i+1} - \underline{E}_p \underline{g}_i). \quad (3.51)$$

We note that, here, the matrix $\underline{N}(\lambda h)$ is of rank one.

Equation (3.48) is of the form

$$\underline{\Phi}_{i+1} = \underline{M} \underline{\Phi}_i + \underline{v}_{i+1}, \quad \underline{M} \equiv \underline{N}(\lambda h). \quad (3.52)$$

which (as noted earlier) is commonplace in the matrix analysis of numerical stability of, for example, ordinary differential equations. Equation (3.52) has interest in its own right (Hahn [10, p47], Miller [19]). We observe that (3.52) may be a scalar relation. The stability analysis which follows is devoted to the analysis of relations which can be put into the form (3.52).

Example 3.2(a)

Consider the conventional R-K tableau

$$\begin{array}{c|cc} \frac{1}{2} & \frac{1}{2} & 0 \\ \hline 1 & 1 & 0 \end{array}$$

we find $\underline{d}(\lambda h) = [(1 - \frac{1}{2}\lambda h)^{-1}, (1 + \frac{1}{2}\lambda h)(1 - \frac{1}{2}\lambda h)^{-1}]^T$

and $\underline{N}(\lambda h) = [\underline{Q} | \underline{d}(\lambda h)]$.

3.4.2 Stability Definitions

We now turn to the provision of stability definitions for the analysis of (3.52).

Definition 3.3(a)

A recurrence of the form (3.52) is said to be (block-) stable in the norm $\|\cdot\|$ if the subordinate norm of the amplification matrix \underline{M} satisfies $\|\underline{M}\| < 1$, and the stability is strict (the method is "contractive") if $\|\underline{M}\| < L < 1$.

The above definition requires a choice of norm. We can introduce new definitions which rely on the location of eigenvalues of \underline{M} . First note that a matrix is said to be of class M [23] if and only if its eigenvalues having largest modulus are semi-simple. (An eigenvalue μ is semi-simple if the number of linearly independent eigenvectors corresponding to μ is equal to the algebraic multiplicity of μ).

Definition 3.3(b)

An arbitrary square matrix \underline{M} will be called strictly stable if $\rho(\underline{M}) < 1$ where $\rho(\underline{M})$ is the spectral radius. A matrix \underline{M} will be called stable if and only if either it is strictly stable or it is of class M and $\rho(\underline{M}) = 1$, (c.f. [27, p.265]).

The stability of a recurrence relation will be related to the stability of its amplification matrix \tilde{M} .

Observe that the spectral radius $\rho(\tilde{M}) \leq ||\tilde{M}||$ of the amplification matrix \tilde{M} satisfies $\rho(\tilde{M}) < 1$ if and only if there exists some subordinate norm with $||\tilde{M}|| < 1$. Suppose now that \tilde{M} is class M (a rank-one matrix is of class M); then $\rho(\tilde{M}) = 1$ if and only if there exists a subordinate norm with $||\tilde{M}|| = 1$. Observe that since $\tilde{M} \equiv \tilde{M}(\lambda h)$ in our applications, the subordinate norm in the latter statement will depend on λh .

Remark

We pause to emphasise how the study of stability of the solution of the integral equation (3.1) is mirrored by studying the effects of perturbations on recurrence relations of the form (3.52). For the basic test equation (3.3) we know that a constant change δ in $g(x)$ results in a change $\epsilon(x) = \delta \exp(\lambda x)$ in $f(x)$. If and only if $\text{Re}(\lambda) < 0$, $\epsilon(x)$ is bounded and (3.3) is stable; if and only if $\text{Re}(\lambda) < 0$, $\lim_{x \rightarrow \infty} \epsilon(x) = 0$ and (3.3) is asymptotically stable.

Definition 3.4 Equation (3.3) is stable if $\text{Re}(\lambda) < 0$ and asymptotically stable if $\text{Re}(\lambda) < 0$.

On the other hand, when equations (3.28) are applied to (3.3) we obtain (3.40) and we seek the effect of perturbations in the values of g_j on the values \tilde{f}_j . Structure in the weights Ω_{jk} enables us to derive a finite term recurrence relation of the form (3.52) where the components of the vectors $\tilde{\Phi}_i$ are successive values of \tilde{f}_j . A perturbation ϵ in $\tilde{\Phi}_0$ results in perturbations $\tilde{M}^k \epsilon$ in $\tilde{\Phi}_k$ and the recurrence for the vectors $\tilde{\Phi}_k$ is damped if and only if $\lim_{k \rightarrow \infty} ||\tilde{M}^k|| = 0$. For any subordinate matrix norm, $\lim_{k \rightarrow \infty} ||\tilde{M}^k||^{1/k} = \rho(\tilde{M})$. Thus the method is damped if $\rho(\tilde{M}) < 1$. However, if $\rho(\tilde{M}) = 1$ and also \tilde{M} is of class M then the perturbations are

bounded.

Applying definitions 3.3 we have the following result.

Theorem 3.3

For given λh , the matrix $\underline{N}(\lambda h)$ is stable, and the recurrence (3.48) is block-stable in some norm, if and only if $\rho(\underline{N}(\lambda h)) \equiv |e_{\underline{p}}^T \underline{d}(\lambda h)| < 1$. The recurrence (3.48) is block-stable in the ℓ_∞ -norm if and only if $\|\underline{N}(\lambda h)\|_\infty \equiv \|\underline{d}(\lambda h)\|_\infty < 1$ and is block-stable in the ℓ_1 -norm if and only if $\|\underline{N}(\lambda h)\|_1 \equiv \|\underline{d}(\lambda h)\|_1 < 1$.

Proof. The proof is straightforward.

3.4.3 Additional Definitions

We shall need some additional definitions later.

It frequently happens that recurrence relations occurring in practice have the form [7]

$$\sum_{\ell=0}^m \underline{x}_\ell(\lambda h) \underline{x}_{n+1-\ell} = \underline{\gamma}_{n+1}, \det[\underline{x}_0(\lambda h)] \neq 0. \quad (3.53)$$

(The vectors and matrices in this relation may reduce to scalars.)

We obtain a relation of the form (3.52) on setting

$$\underline{\Phi}_{n+1} = [\underline{x}_{n+1}^T, \underline{x}_n^T, \dots, \underline{x}_{n-m}^T]^T$$

and

$$\underline{M}(\lambda h) = \begin{bmatrix} -\underline{x}_0^{-1} \underline{x}_1 & -\underline{x}_0^{-1} \underline{x}_2 & \dots & -\underline{x}_0^{-1} \underline{x}_m \\ \underline{I} & \underline{0} & \dots & \underline{0} \\ \underline{0} & \underline{I} & & \underline{0} \\ \vdots & & & \vdots \\ \underline{0} & \dots & \underline{0} & \underline{I} & \underline{0} \end{bmatrix}, \quad (3.54)$$

where $\underline{x}_\ell \equiv \underline{x}_\ell(\lambda h)$. The recurrence relation will therefore be called stable when the recurrence for the vectors $\{\underline{\Phi}_n\}$ is stable:

Definition 3.5

A recurrence (3.53) will be called (strictly) stable if and only if the matrix (3.54) is (strictly) stable.

To assist in detecting whether the matrix $\tilde{M}(\lambda h)$ is of class M (and the nature of its eigenvalues) we observe the following result.

Lemma 3.1

Suppose that $\tilde{z} = [\tilde{z}_1^T, \tilde{z}_2^T, \dots, \tilde{z}_m^T]^T$ and $\tilde{M}(\lambda h)\tilde{z} = \mu\tilde{z}$; then $\tilde{z} = [\mu^{m-1}\tilde{\zeta}^T, \mu^{m-2}\tilde{\zeta}^T, \dots, \tilde{\zeta}^T]^T$ where

$$\left[\sum_{\ell=0}^m \tilde{X}_\ell(\lambda h) \mu^{m-\ell} \right] \tilde{\zeta} = 0, \quad (3.55)$$

and conversely. Thus an eigenvalue μ of $\tilde{M}(\lambda h)$ is semi-simple (that is, the corresponding Jordan block is diagonal) if and only if μ is a semi-simple zero of the "auxiliary polynomial"

$\det \left[\sum_{\ell=0}^m \tilde{X}_\ell(\lambda h) \mu^{m-\ell} \right]$ in the sense that exactly ν linearly independent $\tilde{\zeta}$ satisfy $\left\{ \sum_{\ell=0}^m \tilde{X}_\ell(\lambda h) \mu^{m-\ell} \right\} \tilde{\zeta} = 0$ where ν is the multiplicity of μ as a root of the auxiliary polynomial.

When the matrices \tilde{X}_ℓ reduce to scalars it can be seen that a multiple root cannot be semi-simple. In this case $\tilde{M}(\lambda h)$ is of class M if and only if the roots of largest modulus of the auxiliary polynomial are simple.

The introduction of the spectral radius of the matrix \tilde{M} of (3.54) may be considered somewhat artificial in a discussion of the stability of (3.53). To clarify, we observe that if the vectors

$\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \dots$ are defined by (3.53) and perturbations $\tilde{\xi}_0, \tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_{m-1}$ are made in the "starting vectors" $\tilde{x}_0, \dots, \tilde{x}_{m-1}$, then the resultant changes $\tilde{\xi}_m, \tilde{\xi}_{m+1}, \dots$ in $\tilde{x}_m, \tilde{x}_{m+1}$ satisfy the homogeneous version of (3.53), that is $\sum_{\ell=0}^m \tilde{X}_\ell \tilde{\xi}_{n+1-\ell} = 0$ for $n = m, m+1, \dots$. The choice $\tilde{\xi}_r = \mu^r \tilde{\zeta}$ ($r = 0, 1, \dots, m-1$) where μ is

an eigenvalue of \underline{M} and where $\underline{z} = [\mu^{m-1}\underline{\xi}^T, \mu^{m-2}\underline{\xi}^T, \dots, \underline{\xi}^T]^T$ is the corresponding eigenvector, yields $\underline{\xi}_r = \mu^r \underline{\xi} (r = m, m+1, \dots)$. Thus $\rho(\underline{M}) = \max |\mu|$ gives real insight into the possible growth of perturbations in $\underline{x}_r (r \geq m)$ resulting from certain perturbations in the "starting vectors" $\underline{x}_0, \underline{x}_1, \dots, \underline{x}_{m-1}$. We return to a similar aspect of our theory at the end of §3.4.4.

Since stability of recurrence relations is related to the location of zeros of $\det \left\{ \sum_{\ell=0}^m \underline{x}_{\ell}(\lambda h) \mu^{m-\ell} \right\}$ we recall the following definitions.

Definition 3.6

A polynomial is said to be a von Neumann polynomial if its zeros lie in or on the closed unit-disk centred on the origin, and its zeros of modulus unity are semi-simple. A polynomial is Schur if it is a von Neumann polynomial with no zeros of modulus unity.

3.4.4 Full-step stability

Whilst analysis of (3.48) is sufficient for internal stability (see later, §3.4.5) we can in the present case obtain a scalar recurrence as follows. From (3.48) we find, on using (3.49) and (3.50),

$$\underline{e}_{\underline{p}}^T \varphi_{i+1} = \underline{e}_{\underline{p}}^T (\underline{I} - \lambda h \underline{A})^{-1} \underline{e}_{\underline{p}}^T \varphi_i + \underline{e}_{\underline{p}}^T \nu_{i+1}$$

or, since $\underline{e}_{\underline{p}}^T \varphi_i = \tilde{f}(ih)$,

$$\tilde{f}((i+1)h) = \hat{\mu}(\lambda h) \tilde{f}(ih) + \nu_{i+1} \quad (3.56)$$

where $\hat{\mu}(\lambda h) = \underline{e}_{\underline{p}}^T \underline{d}(\lambda h)$, and $\nu_{i+1} = \underline{e}_{\underline{p}}^T \nu_{i+1}$. Here,

$$\hat{\mu} \equiv \hat{\mu}(\lambda h) = \underline{e}_{\underline{p}}^T (\underline{I} - \lambda h \underline{A})^{-1} \underline{e}_{\underline{p}}, \quad (3.57)$$

and since $\hat{\mu} \underline{e}_{\underline{p}}^T = \underline{e}_{\underline{p}}^T \underline{N}(\lambda h)$, $\hat{\mu}$ is the eigenvalue of $\underline{N}(\lambda h)$ which does not vanish identically. The scalar recurrence (3.56) is stable if and

only if $|\hat{\mu}| \leq 1$, that is $\rho(\tilde{N}(\lambda h)) \leq 1$.

Definition 3.7

A Runge-Kutta method defined by (3.28) and (3.31) displays full-step stability (when applied to a given test equation) if and only if there exists a stable recurrence between vectors where components are the values $\tilde{f}(ih) \equiv \tilde{f}_{i(p+1)}$, $i \geq n_0$.

Remark: The relation between values $\tilde{f}(ih)$ may be of the form (3.53) where the components of the vectors \tilde{x}_0 are values $\tilde{f}(ih)$.

In view of the above, we state the following result.

Theorem 3.4

The extended Runge-Kutta method applied to (3.39) displays full-step stability if and only if $\tilde{N}(\lambda h)$ is stable, that is, if and only if $|\hat{\mu}(\lambda h)| \leq 1$ where $\hat{\mu}(\lambda h)$ is defined by (3.57).

The analysis of extended methods follows similar lines to the analysis of Runge-Kutta methods for $f'(x) = \lambda f(x)$. The notion of block-stability discussed below, provides a tool for the analysis of concepts of internal stability (cf. [13]).

Observe that $\hat{\mu}(\lambda h)$, expressed here as an inner-product $e_{\tilde{p}}^T \tilde{d}(\lambda h)$, can be expressed as a ratio of determinants. We see that $\det \left[\mu(I - \lambda h \tilde{A}) - e_{\tilde{p}} e_{\tilde{p}}^T \right]$ is a polynomial of degree $p+1$ in μ with non-zero root $\hat{\mu}$ and we find

$$\hat{\mu} = \det \begin{bmatrix} 1 - \lambda h A_{00} & -\lambda h A_{01} & \dots & -\lambda h A_{0,p-1} & 1 \\ -\lambda h A_{10} & 1 - \lambda h A_{11} & & & \\ \vdots & & & & \\ \vdots & & & 1 - \lambda h A_{p-1,p-1} & 1 \\ -\lambda h A_{p0} & -\lambda h A_{p1} & \dots & -\lambda h A_{p,p-1} & 1 \end{bmatrix} \div \det [\tilde{I} - \lambda h \tilde{A}].$$

on examining the coefficient of the polynomial, for example. (For "economized versions" referred to in §2.3.4, with $A_{0r} = 0$, $r = 0, 1, \dots, p$ the order can be reduced by one.)

3.4.5 Block stability

With the analysis now developed we can consider a discussion of A-stability.

Definitions 3.8

The region of (block) stability, in some norm, of a recurrence (3.52), where $\underline{M} \equiv \underline{M}(\lambda h)$, is the region of values of $\lambda h \in \mathbb{C}$ such that $\|\underline{M}(\lambda h)\| < 1$, in the subordinate matrix norm. The region of strict stability is the region of values $\lambda h \in \mathbb{C}$ for which $\|\underline{M}(\lambda h)\| < 1$. A recurrence (3.52) is A-stable, in a certain norm, if its region of strict stability includes the region $-\infty < \text{Re}(\lambda h) < 0$. A matrix $\underline{M}(\lambda h)$ will be said to be A-contractive if $\rho(\underline{M}(\lambda h)) < 1$ whenever $-\infty < \text{Re}(\lambda h) < 0$. Strict stability of (3.52) is analogous to (asymptotic) stability of (3.39).

We state, without proof, the following result.

Theorem 3.5

Let the Runge-Kutta tableau (2.33) define a method for an initial-value problem in ordinary differential equations, and let the latter method be applied to the equation $f'(x) = \lambda f(x)$, $f(0)$ given. Then the approximate solution values define vectors satisfying a relation of the form (3.48) with an appropriate choice of vector \underline{v}_{i+1} . The region of stability of the method is identical to the region of stability of the extended Runge-Kutta method applied to

$$f(x) - \lambda \int_0^x f(y) dy = g(x). \quad (\text{See Stetter [25, pp131, 174] and observe}$$

that $\underline{e}_{p\sim}^T \underline{d}(\lambda h)$ in the "growth function" of [25] assuming (2.33)

is a conventional R-K array). In particular if the Runge-Kutta

method is A-stable the recurrence (3.48) is A-stable and the matrix $\tilde{N}(\lambda h)$ is A-contractive.

The following example serves, inter alia, to show that requesting stability (of (3.48)) in a prescribed norm may be too severe.

Example 3.2(b)

Given the conventional R-K tableau of Example 3.2(a)

$$\begin{array}{c|cc} \frac{1}{2} & \frac{1}{2} & 0 \\ \hline 1 & 1 & 0 \end{array}$$

We know that $\tilde{d}(\lambda h) = [(1 - \frac{1}{2}\lambda h)^{-1}, (1 + \frac{1}{2}\lambda h)(1 - \frac{1}{2}\lambda h)^{-1}]^T$ and $\hat{\mu}(\lambda h) = (1 + \frac{1}{2}\lambda h)(1 - \frac{1}{2}\lambda h)^{-1}$.

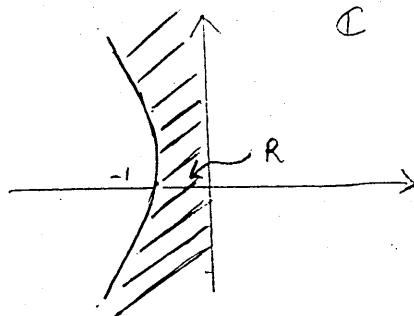
$$\tilde{N}(\lambda h) = \begin{bmatrix} 0 & (1 - \frac{1}{2}\lambda h)^{-1} \\ 0 & (1 + \frac{1}{2}\lambda h)(1 - \frac{1}{2}\lambda h)^{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{d}(\lambda h) \end{bmatrix}.$$

Thus, $\|\tilde{N}(\lambda h)\|_{\infty} = \|\tilde{d}(\lambda h)\|_{\infty} = \max\{|(1 - \frac{1}{2}\lambda h)^{-1}|, |(1 + \frac{1}{2}\lambda h)(1 - \frac{1}{2}\lambda h)^{-1}|\}$

< 1 for all $\operatorname{Re}(\lambda) < 0$ for all h

and $\|\tilde{N}(\lambda h)\|_1 = \|\tilde{d}(\lambda h)\|_1 = \{|(1 - \frac{1}{2}\lambda h)^{-1}| + |(1 + \frac{1}{2}\lambda h)(1 - \frac{1}{2}\lambda h)^{-1}|\}$.

> 1 if $0 > \lambda h > -1$ and $\lambda \in \mathbb{R}$.



$\|\tilde{d}(\lambda h)\|_1 > 1$ if $h\lambda \in \mathbb{R} = \{(\alpha, \beta) : \alpha < 0 \text{ \& } \alpha^2 - \frac{1}{3}\beta^2 < 1\}$.

The recurrence (3.53) is A-stable, and the recurrence (3.48) is A-stable in the uniform norm. Moreover (3.48) is strictly block-stable

in the uniform norm when $\operatorname{Re}(\lambda h) < 0$. In the $\|\cdot\|_1$ norm this feature is lost and $\lambda h = 0$ is not in the region of block stability.

Example 3.3

Consider the Runge-Kutta tableau for an A-stable semi-explicit method of order two [16, p.244].

0		0	
1		$\frac{1}{2}$	$\frac{1}{2}$
<hr/>			
		$\frac{1}{2}$	$\frac{1}{2}$

We find $\underline{d}(\lambda h) = [1, (1+\frac{1}{2}\lambda h)(1-\frac{1}{2}\lambda h)^{-1}, (1+\frac{1}{2}\lambda h)(1-\frac{1}{2}\lambda h)^{-1}]^T$ and $\hat{\mu}(\lambda h) = (1+\frac{1}{2}\lambda h)(1-\frac{1}{2}\lambda h)^{-1}$. Thus, the extended R-K method displays full-step stability since $|\hat{\mu}(\lambda h)| < 1$ when $-\infty < \operatorname{Re}(\lambda h) < 0$.

$$\|\underline{N}(\lambda h)\|_1 = 1 + 2 |(1+\frac{1}{2}\lambda)(1-\frac{1}{2}\lambda h)^{-1}|$$

$$\|\underline{N}(\lambda h)\|_\infty = \max\{1, |(1+\frac{1}{2}\lambda h)(1-\frac{1}{2}\lambda h)^{-1}|\}.$$

When $-\infty < \operatorname{Re}(\lambda h) < 0$ $\|\underline{N}(\lambda h)\|_\infty = 1$ and we have block stability but not strict stability in the uniform norm. In the $\|\cdot\|_1$ norm block-stability does not exist. Once again, we stress that the choice of norm is significant in discussing stability and regions of stability.

Remark The final observation of the preceding example shows that the choice of norm is significant in discussing stability and regions of stability and it also affects the practical relevance of the theory. In the present case it is easily argued that $\hat{\mu}(\lambda h)$ is the significant factor affecting stability, first in respect of its rôle in full-step stability but also from the following reasoning. We suppose that $\underline{\Phi}_{k+1} = \underline{N}(\lambda h)\underline{\Phi}_k + \underline{\gamma}_{k+1}$ and that $\underline{\Phi}_0$ is perturbed by the addition of $\underline{\xi}_0$. The consequent change $\underline{\xi}_k$ in $\underline{\Phi}_k$ is $(\underline{N}(\lambda h))^k \underline{\xi}_0$ and $\|\underline{\xi}_k\|_\infty \leq \|(\underline{N}(\lambda h))^k\|_\infty \|\underline{\xi}_0\|_\infty \leq (\|\underline{N}(\lambda h)\|_\infty)^k \|\underline{\xi}_0\|_\infty$. The latter inequality loses much however. With $\underline{N}(\lambda h) = \underline{d}(\lambda h) \underline{e}_p^T$, we find

$\{\tilde{N}(\lambda h)\}^k = \{\hat{\mu}(\lambda h)\}^{k-1} \tilde{N}(\lambda h)$. Thus, $\|\{\tilde{N}(\lambda h)\}^k\| = \|\hat{\mu}(\lambda h)\|^{k-1} \|\tilde{N}(\lambda h)\|$, and $\|\tilde{\xi}_k\| \leq \|\tilde{\xi}_0\|$ if the latter factor is bounded by unity for some norm. In the more general case of a relation (3.52), in which the amplification matrix has no special structure, we may appeal to the definition $\rho(\tilde{M}(\lambda h)) = \lim_{k \rightarrow \infty} \|\{\tilde{M}(\lambda h)\}^k\|^{1/k}$ which is valid for any subordinate norm. Thus, given $\varepsilon > 0$ there exists a corresponding integer $k(\varepsilon)$ such that, for $k \geq k(\varepsilon)$, $\|\{\tilde{M}(\lambda h)\}^k\| \leq (\rho(\tilde{M}(\lambda h)) + \varepsilon)^k$.

3.4.6 Modified extended methods

As noted earlier, in §2.3.4 variants of the extended methods with $\tilde{e} \tilde{a}_p^T$ of (3.32) replaced by $\tilde{e} \tilde{b}_p^T$ may be found in the literature. For this method (3.45) is replaced by

$$(\tilde{I} - \lambda h \tilde{A}) \tilde{\varphi}_{i+1} - \lambda h \tilde{e} \tilde{b}_p^T \sum_{j=1}^i \tilde{\varphi}_j = \tilde{g}_{i+1} \quad (3.58)$$

and the result analogous to (3.47) is

$$(\tilde{I} - \lambda h \tilde{A}) \tilde{\varphi}_{i+1} = (\tilde{E}_p + \lambda h \tilde{e} (\tilde{b}_p^T - \tilde{a}_p^T)) \tilde{\varphi}_i + \tilde{g}_{i+1} - \tilde{E}_p \tilde{g}_i \quad (3.59)$$

Equation (3.59) assumes the form (3.52) with an amplification matrix

$$\tilde{N}_*(\lambda h) = \tilde{d}(\lambda h) \left[\tilde{e}_p^T (\tilde{I} - \lambda h \tilde{A}) + \lambda h \tilde{b}_p^T \right],$$

on applying $(\tilde{I} - \lambda h \tilde{A})^{-1}$ and writing $\tilde{a}_p^T = \tilde{e}_p^T \tilde{A}$. The matrix $\tilde{N}_*(\lambda h)$ is of rank one and its non-trivial eigenvalue is expressible as

$$\hat{\mu}_*(\lambda h) = 1 + \lambda h \tilde{b}_p^T (\tilde{I} - \lambda h \tilde{A})^{-1} \tilde{e}_p. \quad (3.60)$$

When $\tilde{b}_p^T \neq \tilde{a}_p^T$ it is more difficult to derive the exact analogue of the scalar relation (3.56).

Paul Wolkenfelt has observed that we may proceed in the following way.

Equation (3.58) becomes, on applying $\tilde{e}_p^T (\tilde{I} - \lambda h \tilde{A})^{-1}$,

$$\tilde{e}_p^T \tilde{\varphi}_{i+1} = \tilde{e}_p^T (\tilde{I} - \lambda h \tilde{A})^{-1} \tilde{e} - \lambda h \tilde{b}_p^T \sum_{j=1}^i \tilde{\varphi}_j + \tilde{e}_p^T (\tilde{I} - \lambda h \tilde{A})^{-1} \tilde{g}_{i+1} \quad (3.61)$$

or, since $\underline{e}_p^T \underline{\varphi}_i = \tilde{f}(ih)$,

$$\tilde{f}_{i+1} = \hat{\mu}(\lambda h) \lambda h \underline{b}_p^T \sum_{j=1}^i \underline{\varphi}_j + \underline{e}_p^T \underline{w}_{i+1} \quad (3.62)$$

$$\text{where } \underline{w}_{i+1} = (\underline{I} - \lambda h \underline{A})^{-1} \underline{g}_{i+1}. \quad (3.63)$$

We observe that, on using (3.58),

$$\underline{b}_p^T \left(\sum_{j=1}^i \underline{\varphi}_j - \sum_{j=1}^{i-1} \underline{\varphi}_j \right) = \underline{b}_p^T \underline{\varphi}_i = \underline{b}_p^T (\underline{I} - \lambda h \underline{A})^{-1} \underline{e} \lambda h \underline{b}_p^T \sum_{j=1}^{i-1} \underline{\varphi}_j + \underline{b}_p^T \underline{w}_i. \quad (3.64)$$

$$\text{Thus, } \underline{b}_p^T \sum_{j=1}^i \underline{\varphi}_j = [1 + \lambda h \underline{b}_p^T (\underline{I} - \lambda h \underline{A})^{-1} \underline{e}] \underline{b}_p^T \sum_{j=1}^{i-1} \underline{\varphi}_j + \underline{b}_p^T \underline{w}_i. \quad (3.65)$$

On applying $\hat{\mu}(\lambda h) \lambda h$ we obtain

$$\hat{\mu}(\lambda h) \lambda h \underline{b}_p^T \sum_{j=1}^i \underline{\varphi}_j = \mu_*(\lambda h) \hat{\mu}(\lambda h) \lambda h \underline{b}_p^T \sum_{j=1}^{i-1} \underline{\varphi}_j + \hat{\mu}(\lambda h) \lambda h \underline{b}_p^T \underline{w}_i. \quad (3.66)$$

From (3.62), and using (3.66),

$$\tilde{f}_{i+1} - \underline{e}_p^T \underline{w}_{i+1} = \mu_*(\lambda h) [\tilde{f}_i - \underline{e}_p^T \underline{w}_i] + \hat{\mu}(\lambda h) \lambda h \underline{b}_p^T \underline{w}_i. \quad (3.67)$$

Thus,

$$\tilde{f}_{i+1} = \mu_*(\lambda h) \tilde{f}_i + [\underline{e}_p^T \underline{w}_{i+1} - \mu_*(\lambda h) \underline{e}_p^T \underline{w}_i + \hat{\mu}(\lambda h) \lambda h \underline{b}_p^T \underline{w}_i]. \quad (3.68)$$

Theorem 3.6

The recurrence (3.59) associated with the modified extended method is block-stable in some norm (depending on λh) if and only if

$$\hat{\mu}_*(\lambda h) \equiv 1 + \lambda h \underline{b}_p^T (\underline{I} - \lambda h \underline{A})^{-1} \underline{e}$$

lies in or on the unit circle centered on the origin.

Example 3.4

Consider the third-order Runge-Kutta tableau

0	0	0	0	0
$\frac{1}{3}$	$\frac{1}{3}$	0	0	0
$\frac{2}{3}$	0	$\frac{2}{3}$	0	0
...
1	$\frac{1}{4}$	0	$\frac{3}{4}$	0

with which we may associate $\tilde{b}^T = \left[\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8} \right]$. We find

$$\tilde{d}(\lambda h) = \left[1, 1 + \frac{1}{3}\lambda h, 1 + \frac{2}{3}\lambda h + \frac{2}{9}\lambda^2 h^2, 1 + \lambda h + \frac{1}{2}\lambda^2 h^2, + \frac{1}{6}\lambda^3 h^3 \right]^T.$$

Then $\hat{\mu}(\lambda h) = 1 + \lambda h + \frac{1}{2}\lambda^2 h^2 + \frac{1}{6}\lambda^3 h^3$, $\hat{\mu}_*(\lambda h) = 1 + \lambda h + \frac{1}{2}\lambda^2 h^2 +$

$\frac{7}{48}\lambda^3 h^3 + \frac{1}{48}\lambda^4 h^4$. In the $\|\cdot\|_1$ norm and the $\|\cdot\|_\infty$ norm block

stability exists as there is a region of values $\lambda h \in \mathbb{C}$ such that

$|\mu_*(\lambda h)| < 1$. ($|\mu_*(\lambda h)| = 1$ when $\lambda h = 0, -4, -\frac{3}{2}(1 \pm i\sqrt{\frac{13}{3}})$;
 $\mu_*(\lambda h) = -1$ has 4 complex roots. Region of stability is $[-4, 0]$)

We note that the result implies a lower-order accuracy in the modified

method. (Consider $\exp(\lambda h) - \hat{\mu}_*(\lambda h)$.) The modified method in this

example requires more effort per step than the extended method.

3.5 Mixed Runge-Kutta methods with reducible rules

Amongst the class of mixed Runge-Kutta methods are those employing reducible quadrature rules Q (2.3) satisfying (2.22). Before

proceeding with the analysis of such methods it is useful to recall

that the stability polynomial for a linear multi-step method applied to

$f'(x) = \lambda f(x)$ is $\rho(\mu) - \lambda h \sigma(\mu)$. If we suppose that the rules Q are

(ρ, σ) - reducible where ρ, σ are defined by (2.17) then a simple

calculation shows that the "basic test equation" (3.39), resulting in the

particular form of (3.40) under consideration, yields, using (2.22),

$$\sum_{\ell=0}^m \alpha_\ell \tilde{f}((n-\ell)h) = \lambda h \sum_{\ell=0}^m \beta_\ell \tilde{f}((n-\ell)h) + \sum_{\ell=0}^m \alpha_\ell g((n-\ell)h), \quad n \geq n_0. \quad (3.69)$$

This is a finite term recurrence with stability polynomial

$$\rho(\mu) - \lambda h \sigma(\mu).$$

Returning to the mixed Runge-Kutta methods we observe that the structure of the equations for approximate values is derived from

(3.33) and we obtain from (3.40) the scalar equations

$$\tilde{f}_{i(p+1)+1} = \lambda h [\omega_{i0} \tilde{f}_0 + \omega_{i1} \tilde{f}_{p+1} + \dots + \omega_{ii} \tilde{f}_{i(p+1)}] +$$

$$+ \lambda h [A_{00} \tilde{f}_{i(p+1)+1} + A_{01} \tilde{f}_{i(p+1)+2} + \dots + A_{op} \tilde{f}_{(i+1)(p+1)}] + g(\tau_{i(p+1)+1})$$

$$\tilde{f}_{i(p+1)+2} = \lambda h [\omega_{i0} \tilde{f}_0 + \omega_{i1} \tilde{f}_{p+1} + \dots + \omega_{ii} \tilde{f}_{i(p+1)}] +$$

$$+ \lambda h [A_{10} \tilde{f}_{i(p+1)+1} + A_{11} \tilde{f}_{i(p+1)+2} + \dots + A_{1p} \tilde{f}_{(i+1)(p+1)}] + g(\tau_{i(p+1)+2})$$

.

.

.

$$\tilde{f}_{(i+1)(p+1)} = \lambda h [\omega_{i0} \tilde{f}_0 + \omega_{i1} \tilde{f}_{p+1} + \dots + \omega_{ii} \tilde{f}_{i(p+1)}] +$$

$$+ \lambda h [A_{p0} \tilde{f}_{i(p+1)+1} + A_{p1} \tilde{f}_{i(p+1)+2} + \dots + A_{pp} \tilde{f}_{(i+1)(p+1)}] + g(\tau_{(i+1)(p+1)}) \quad (3.70)$$

We define $\varphi_1, \varphi_2, \varphi_3 \dots$ by (3.41) and set $\varphi_0 = g(0) \underline{e}$.

Then, with $\underline{E}_p = \underline{e} \underline{e}^T$, we have

$$(\underline{I} - \lambda h \underline{A}) \varphi_{i+1} - \lambda h \sum_{k=0}^i \omega_{ik} \underline{E}_p \varphi_k = \underline{g}_{i+1}. \quad (3.71)$$

It follows that, when $(\underline{I} - \lambda h \underline{A})^{-1}$ exists

$$\varphi_{i+1} - \lambda h \sum_{k=0}^i \omega_{ik} \underline{N}(\lambda h) \varphi_k = \underline{\gamma}_{i+1} \quad (3.72)$$

where

$$\underline{\gamma}_{i+1} = (\underline{I} - \lambda h \underline{A})^{-1} \underline{g}_{i+1} \quad \text{and} \quad \underline{N}(\lambda h)$$

is defined by (3.49).

With the aid of (2.22) we find on setting $i = n - \ell$ in (3.72),

multiplying by α_ℓ and summing over ℓ

$$\sum_{\ell=0}^m \alpha_\ell \varphi_{n+1-\ell} - \lambda h \sum_{\ell=0}^m \beta_\ell \underline{N}(\lambda h) \varphi_{n-\ell} = \hat{\underline{\gamma}}_{n+1}, \quad (3.73)$$

$$\text{where } \hat{\underline{\gamma}}_{n+1} = \sum_{\ell=0}^m \alpha_\ell \underline{\gamma}_{n+1-\ell}.$$

Since this type of argument is repeatedly used below, we pause to clarify it. In deducing (3.73) we obtain from (3.72) the result

$$\sum_{\ell=0}^m \alpha_{\ell} \varphi_{n+1-\ell} - \lambda h \sum_{\ell=0}^m \alpha_{\ell} \sum_{k=0}^{n-\ell} \omega_{n-\ell, k} \varphi_k = \sum_{\ell=0}^m \alpha_{\ell} \gamma_{n+1-\ell}.$$

The argument is more clearly seen on writing $\omega_{ik} = 0$ for $k > i$ so that the second term on the left is

$$\lambda h N(\lambda h) \sum_{\ell=0}^m \alpha_{\ell} \sum_{k \geq 0} \omega_{n-\ell, k} \varphi_k = \lambda h N(\lambda h) \sum_{k \geq 0} \left\{ \sum_{\ell=0}^n \alpha_{\ell} \omega_{n-\ell, k} \right\} \varphi_k.$$

On using (2.22) we obtain

$$\lambda h N(\lambda h) \sum_{k \geq 0} \beta_{n-k} \varphi_k = \lambda h N(\lambda h) \sum_{\ell=0}^m \beta_{\ell} \varphi_{n-\ell}.$$

As in the analysis leading of §3.4.3 the relation (3.73) can be expressed as

$$\begin{bmatrix} \varphi_{n+1} \\ \varphi_n \\ \vdots \\ \varphi_{n+1-m} \end{bmatrix} = \begin{bmatrix} \alpha_0 I & \alpha_1 I & \dots & \alpha_m I \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix}^{-1} \lambda h \begin{bmatrix} \beta_0 N(\lambda h) & \beta_1 N(\lambda h) & \dots & \beta_m N(\lambda h) \\ & I & & \\ & & \ddots & \\ & & & I & 0 \end{bmatrix} \begin{bmatrix} \varphi_n \\ \varphi_{n-1} \\ \vdots \\ \varphi_{n-m} \end{bmatrix} + [\hat{\gamma}_{n+1}^T, 0^T, \dots, 0^T]^T, \quad (3.74)$$

Writing $\psi_{n+1}^* = [\varphi_{n+1}^T, \varphi_n^T, \dots, \varphi_{n+1-m}^T]^T$ the relation (3.74) may be expressed in the form

$$\psi_{n+1}^* = \hat{M} \psi_n^* + \Gamma_{n+1}$$

where the amplification matrix $\hat{M} \equiv \hat{M}(\lambda h)$ here has as its eigenvalues the roots of the auxiliary equation

$$\det \left[\sum_{\ell=0}^m \{ \alpha_{\ell} \mu I - \lambda h \beta_{\ell} N(\lambda h) \} \mu^{m-\ell} \right] = 0. \quad (3.75)$$

We shall strengthen the following result later.

Theorem 3.5

The recurrence (3.73) is stable if the roots of (3.75) lie in or on the closed unit disk centered on the origin, and those of modulus unity are semi-simple, that is, if $\det[\mu \rho^*(\mu) \underline{I} - \lambda h \sigma^*(\mu) \underline{N}(\lambda h)]$ is a von Neumann polynomial.

3.5.1 Full-step stability

We turn now to the consideration of full-step stability for the mixed method considered above.

Taking inner-products with \underline{e}_p in (3.73) yields, with

$\underline{N}(\lambda h) = \underline{d}(\lambda h) \underline{e}_p^T$ as in (3.49), the result

$$\sum_{\ell=0}^m \alpha_{\ell} \underline{e}_p^T \varphi_{n+1-\ell} - \lambda h \sum_{\ell=0}^m \beta_{\ell} \underline{e}_p^T \underline{N}(\lambda h) \varphi_{n-\ell} = \sum_{\ell=0}^m \alpha_{\ell} \underline{e}_p^T \gamma_{n+1-\ell}. \quad (3.76)$$

Thus

$$\sum_{\ell=0}^m \alpha_{\ell} \tilde{f}((n+1-\ell)h) - \lambda h \hat{\mu}(\lambda h) \sum_{\ell=0}^m \beta_{\ell} \tilde{f}((n-\ell)h) = \xi_{n+1} \quad (3.77)$$

where ξ_{n+1} denotes the right-hand term in (3.76), and

$\hat{\mu}(\lambda h) = \underline{e}_p^T \underline{d}(\lambda h)$, as in (3.57). Writing $\varphi_{n+1}^* = [f((n+1)h),$

$\tilde{f}(nh), \dots, \tilde{f}((n+1-m)h)]^T$ we may express (3.77) in the form (3.52)

where the characteristic polynomial of the amplification matrix is

$$\sum_{\ell=0}^m \{\alpha_{\ell} \mu - \lambda h \hat{\mu}(\lambda h) \beta_{\ell}\} \mu^{m-\ell}. \quad (3.78)$$

We have established the following result.

Theorem 3.6

Let $\rho^*(\mu) = \sum_{\ell=0}^m \alpha_{\ell} \mu^{m-\ell}$, $\sigma^*(\mu) = \sum_{\ell=0}^m \beta_{\ell} \mu^{m-\ell}$ and let the rules (2.3) be reducible to the linear multistep method whose first and second characteristic polynomials are $\rho^*(\mu)$, $\sigma^*(\mu)$. Then the mixed Runge-Kutta method displays full-step stability if and only if

$$\mu \rho^*(\mu) - \lambda h \hat{\mu}(\lambda h) \sigma^*(\mu) \quad (3.79)$$

is a von Neumann polynomial. The stability is strict if (3.79) is Schur.

Theorem 3.7 prompts re-examination of (3.75) to relate it to (3.79) under standard assumptions [25, pp.188, 206].

3.5.2 Block Stability

From the relation (3.75) we can establish the following results which are a consequence of the two previous theorems.

Theorem 3.7

Suppose the conditions of Theorem 3.6 prevail and in addition $\rho^*(\mu)$ is a von Neumann polynomial with $\rho^*(1) = 0$. Then (3.73) is block-stable if and only if the method exhibits full-step stability.

Proof. The value μ' is a root of (3.75) if and only if, for some $\xi \neq 0$, we have

$$\lambda h \sigma^*(\mu') \underset{\sim}{N}(\lambda h) \underset{\sim}{\xi} = \mu' \rho^*(\mu') \underset{\sim}{\xi}.$$

If ρ^* and σ^* have no common factors either $\mu' = 0$ and $\underset{\sim}{N}(\lambda h) \underset{\sim}{\xi} = 0$ or $\sigma^*(0) = 0$

$$\underset{\sim}{N}(\lambda h) \underset{\sim}{\xi} = \mu^* \underset{\sim}{\xi} \tag{3.80}$$

where

$$\mu^* = \mu' \rho^*(\mu') / (\lambda h \sigma^*(\mu')). \tag{3.81}$$

For (3.80) to be satisfied, $(\mu^*, \underset{\sim}{\xi})$ must be an eigenpair of $\underset{\sim}{N}(\lambda h)$ and hence either $\mu^* = 0$, whence $\rho^*(\mu') = 0$ or $\mu^* = \hat{\mu}(\lambda h)$ whence μ' must be a zero of the polynomial (3.79) which occurs in the statement of Theorem 3.6.

If μ' is a root of (3.79) then $\underset{\sim}{\xi}$ is necessarily a multiple of $\underset{\sim}{d}(\lambda h)$ and μ' is semi-simple only if it is simple. (If μ' is a zero of $\rho^*(\mu)$ then $\underset{\sim}{\xi}$ is any vector in a p-dimensional subspace of vectors annihilated by $\underset{\sim}{N}(\lambda h)$. We expect to find $\rho^*(\mu') = 0$, for some μ' with $|\mu'| = 1$ and if μ' is a zero of (3.79) then there exist two linearly independent vectors $\underset{\sim}{\xi}$ corresponding to this value.)

Example 3.5

Consider the mixed Runge-Kutta method employing the repeated trapezium rule (Example 2.7) and the Runge-Kutta tableau $\begin{array}{c|ccc} \frac{1}{2} & \frac{1}{2} & & 0 \\ \hline 1 & 1 & \cdots & 0 \end{array}$ which yields $\hat{\mu}(\lambda h) = (1 + \frac{1}{2}\lambda h) / (1 - \frac{1}{2}\lambda h)$ (see Example 3.2). The polynomials $\rho^*(\mu)$, $\sigma^*(\mu)$ are determined by the coefficients given in Example 2.7 and the polynomial (3.75) is

$$\mu(\mu-1) - \frac{1}{2}\lambda h(1+\frac{1}{2}\lambda h)(1-\frac{1}{2}\lambda h)^{-1}(\mu+1).$$

The zeros of this polynomial are

$$\mu_1 = -\frac{\lambda h}{2}, \quad \mu_2 = \frac{1+\frac{1}{2}\lambda h}{1-\frac{1}{2}\lambda h}.$$

The method exhibits full-step stability if and only if $\text{Re}(\lambda h) < 0$ and $|\lambda h| < 2$.

In view of the preceding example we may state the following theorem, which is of some significance.

Theorem 3.8

A mixed method based on an A-stable reducible quadrature rule and an A-stable Runge-Kutta tableau need not exhibit full-step A-stability.

3.5.3 Economized versions

The previous theorem is disturbing when A-stability is a desirable feature, since mixed quadrature - Runge-Kutta methods consume less effort than their extended counterparts. However, the mixed methods can sometimes be modified to obtain full step A-stability. We consider only the "economized version" of the mixed method, applicable where $\theta_0 = 0$, and recall from §2.3.4 $\tilde{f}(ih) = \tilde{f}_i(p+1) = \tilde{f}_{i(p+1)+1}$, that is,

$$\tilde{e}_p^T \tilde{\varphi}_i = \tilde{e}_0^T \tilde{\varphi}_{i+1} \quad (3.82)$$

The first of the scalar equations in (3.70), using (3.82), may be written in vector form as

$$\underline{e}_0 \underline{e}_0^T \underline{\varphi}_{i+1} = \underline{e}_0 \underline{e}_p^T \underline{\varphi}_i = \lambda h \sum_{k=0}^i \omega_{ik} \underline{e}_0 \underline{e}_p^T \underline{\varphi}_k + \lambda h \underline{e}_0 \underline{e}_0^T \underline{A} \underline{\varphi}_{i+1} + \underline{e}_0 \underline{e}_0^T \underline{g}_{i+1} \quad (3.83)$$

Subtracting (3.83) from (3.71) we obtain

$$(\underline{I} - \lambda h \underline{\hat{A}}) \underline{\varphi}_{i+1} - \lambda h \sum_{k=0}^i \omega_{ik} (\underline{e} - \underline{e}_0) \underline{e}_p^T \underline{\varphi}_k = \underline{e}_0 \underline{e}_p^T \underline{\varphi}_i + [\underline{I} - \underline{e}_0 \underline{e}_0^T] \underline{g}_{i+1}$$

where

$$\underline{\hat{A}} = [\underline{I} - \underline{e}_0 \underline{e}_0^T] \underline{A}. \quad (3.84)$$

In consequence,

$$\underline{\varphi}_{i+1} - \lambda h \sum_{k=0}^i \omega_{ik} [\underline{N}_*(\lambda h) - \underline{N}_0(\lambda h)] \underline{\varphi}_k = \underline{N}_0(\lambda h) \underline{\varphi}_i + \underline{\gamma}_{i+1} \quad (3.85)$$

where

$$\underline{N}_0(\lambda h) = [\underline{I} - \lambda h \underline{\hat{A}}]^{-1} \underline{e}_0 \underline{e}_p^T$$

$$\underline{N}_*(\lambda h) = [\underline{I} - \lambda h \underline{\hat{A}}]^{-1} \underline{e} \underline{e}_p^T$$

$$\text{and} \quad \underline{\gamma}_{i+1} = [\underline{I} - \lambda h \underline{\hat{A}}]^{-1} [\underline{I} - \underline{e}_0 \underline{e}_0^T] \underline{g}_{i+1}. \quad \text{Thus}$$

$$\begin{aligned} \sum_{\ell=0}^m \alpha_{\ell} (\underline{\varphi}_{n+1-\ell} - \underline{N}_0(\lambda h) \underline{\varphi}_{n-\ell}) - \lambda h \sum_{\ell=0}^m \beta_{\ell} [\underline{N}_*(\lambda h) - \underline{N}_0(\lambda h)] \underline{\varphi}_{n-\ell} \\ = \sum_{\ell=0}^m \alpha_{\ell} \underline{\gamma}_{n+\ell-1}. \end{aligned} \quad (3.86)$$

If we consider full-step stability we have, on taking the inner products of (3.86) with \underline{e}_p , the result

$$\begin{aligned} \sum_{\ell=0}^m \alpha_{\ell} \{ \tilde{f}((n+1-\ell)h) - \mu_0(\lambda h) \tilde{f}((n-\ell)h) \} \\ - \lambda h \sum_{\ell=0}^m \beta_{\ell} \{ \mu_*(\lambda h) - \mu_0(\lambda h) \} \tilde{f}((n-\ell)h) \\ = \sum_{\ell=0}^m \alpha_{\ell} \underline{e}_p^T \underline{\gamma}_{n-\ell+1} \end{aligned} \quad (3.87)$$

where, in the notation of (3.84)

$$\mu_0(\lambda h) = \underset{\sim}{e}_p^T [\underset{\sim}{I} - \lambda h \underset{\sim}{A}]^{-1} \underset{\sim}{e}_0$$

and

(3.88)

$$\mu_*(\lambda h) = \underset{\sim}{e}_p^T [\underset{\sim}{I} - \lambda h \underset{\sim}{A}]^{-1} \underset{\sim}{e}$$

We deduce the following result.

Theorem 3.9

Suppose that in the economized mixed method, the quadrature rules (2.3) are (ρ^*, σ^*) -reducible. Then the method applied to (3.39) displays full-step stability if and only if

$$(\mu - \mu_0(\lambda h)) \rho^*(\mu) - \lambda h (\mu_*(\lambda h) - \mu_0(\lambda h)) \sigma^*(\mu)$$

is a von Neumann polynomial, and the stability is strict if and only if it is Schur.

Example 3.6

Consider the economized version of the method in Example 3.5,

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad (\underset{\sim}{I} - \lambda h \underset{\sim}{A})^{-1} = \begin{bmatrix} 1 & 0 \\ \lambda h & 1 \end{bmatrix}$$

$$\mu_0 = \lambda h, \quad \mu_* = 1 + \lambda h, \quad \rho^*(\mu) = \mu - 1, \quad \sigma^*(\mu) = \frac{1}{2}(\mu + 1).$$

We are interested in the polynomial

$$(\mu - \lambda h)(\mu - 1) - \frac{\lambda}{2} h (\mu + 1)$$

that is,

$$\mu^2 - \left[1 + \frac{3\lambda h}{2}\right] \mu + \frac{\lambda h}{2}.$$

Denoting its zeros by μ_1 and μ_2 , $\mu_1 = \exp(\lambda h) + O(h^3)$ and $\mu_2 = \frac{\lambda h}{2} \exp(-\lambda h) + O(h^3)$. The economized method exhibits full-step stability if and only if $\text{Re}(\lambda h) < 0$ and $|\lambda h| < 1$.

Unfortunately, it is not always true that economized methods have increased regions of stability.

3.6 Mixed quadrature Runge-Kutta methods using block-reducible rules.

Now, we consider the stability of mixed Runge-Kutta rules in which the quadrature rules are block-reducible in the sense of Definition 2.5. Thus we suppose that the weights (ω_{ij}) of the quadrature rules employed in the mixed method can be partitioned into square matrices of order q represented in (2.8) such that for fixed matrices $\{\underline{A}_\ell, \underline{B}_\ell\}_{\ell=0}^m$ (defining $V_{\ell j} = 0$ if $j > \ell$) they satisfy certain conditions (3.38) which are here repeated for convenience.

$$\sum_{\ell=0}^m \underline{A}_\ell \underline{V}_{n+1-\ell, \nu} = \underline{B}_{n+1-\nu}, \quad n \geq n_0 \geq 0$$

where $\underline{A}_\nu = \underline{B}_\nu = \underline{0}$ for $\nu \notin \{0, 1, 2, \dots, m\}$

$$\text{and } \sum_{\ell=0}^m \underline{A}_\ell \underline{\varepsilon} = \underline{0} \quad \text{where } \underline{\varepsilon} = [1, 1, \dots, 1]^T \in \mathbb{R}^q \quad (3.89)$$

Relations (3.89) correspond to generalisations of (2.22). We assume with little loss, that $n_0 = n'_0 = 0$.

3.6.1 Block Stability

(For convenience, below, $\underline{A}_{m+1} = \underline{B}_{m+1} = \underline{B}_{-1} = \underline{0}$.)

We require some additional notation and we set, for $\ell = 0, 1, \dots, m+1$,

$$\hat{\underline{B}}_\ell = \underline{B}_{\ell-1} \underline{J}^\# + \underline{B}_\ell \underline{J} \quad (3.90)$$

where, in partitioned form,

$$\underline{J} = [\underline{\varepsilon}_1, \underline{\varepsilon}_2, \dots, \underline{\varepsilon}_{q-1}, \underline{0}], \quad (3.91a)$$

$$\underline{J}^\# = [0, 0, \dots, 0, \underline{\varepsilon}_0], \quad (3.91b)$$

and $\underline{\varepsilon}_0, \underline{\varepsilon}_1, \dots, \underline{\varepsilon}_{q-1}$ are the successive columns of the identity matrix \underline{I}_q of order q . (We continue to write \underline{I} for the identity of order $p+1$.) Recall the definition of $\hat{\mu}(\lambda h)$ in (3.57). The principal result of this section may now be stated.

Theorem 3.10

Suppose the mixed-Runge-Kutta method employs block-reducible

quadrature formulae satisfying Definition 2.5, and suppose the matrices \hat{B}_ℓ ($\ell = 0, 1, \dots, m+1$) to be defined by (3.90). Then the method displays full-step stability if and only if

$$\det \left[\sum_{\ell=0}^{m+1} (A_\ell - \lambda h \hat{\mu}(\lambda h) \hat{B}_\ell) \mu^{m-\ell+1} \right]$$

is a von Neumann polynomial; the stability is strict if and only if the polynomial is Schur.

We apply the theorem to an example later but now undertake a proof of the theorem. The partitioning of the weights (ω_{ij}) into submatrices \hat{V}_{ij} satisfying (3.89) is not ideal for our discussion. We wish to 'exclude' the weights ω_{i0} before partitioning. We therefore define matrices \hat{V}_{ij} by prescribing the columns

$$\begin{aligned} \hat{V}_{ij} \epsilon_r &= V_{ij} \epsilon_{r+1}, \quad r = 0, 1, \dots, q-2, \\ \hat{V}_{ij} \epsilon_{q-1} &= V_{i, j+1} \epsilon_0 \end{aligned} \quad (3.92)$$

for $j = 0, 1, 2, \dots$,

where $\hat{V}_{ij} = 0$ if $j > i$. Thus,

$$\hat{V}_{ij} = V_{ij} J + V_{i, j+1} J^\# \quad (3.93)$$

Since \hat{V}_{ii} is lower triangular the matrix \hat{V}_{ji} is strictly lower triangular.

We shall employ the notation for a Kronecker (direct) product between matrices.

Definition 3.9

Let \hat{A}, \hat{B} be square matrices of order m, n respectively. Then $\hat{A} \otimes \hat{B}$ is a matrix of order mn whose (r, s) -th submatrix is $A_{rs} \hat{B}$, where A_{rs} is the entry in the r -th row, s -th column of \hat{A} .

Remark

For future use we state the following result:

$$[\hat{A} \otimes \hat{B}] [\hat{C} \otimes \hat{D}] = \hat{A} \hat{C} \otimes \hat{B} \hat{D}. \quad (3.94)$$

We set, for $k = 0, 1, 2, \dots$,

$$\hat{\varphi}_k = [\varphi_{kq+1}^T, \varphi_{kq+2}^T, \dots, \varphi_{(k+1)q}^T]^T \quad (k = 0, 1, 2, \dots), \quad (3.95)$$

where φ_j is defined by (3.41). Equations (3.71) can be re-written as

$$(I - \lambda h A) \varphi_{kq+1} - \lambda h \sum_{\ell=1}^{kq} \omega_{kq, \ell} E_p \varphi_\ell = g_{kq+1} + \lambda h \omega_{kq, 0} E_p \varphi_0.$$

On reflection (see Example 3.7 below) we find that the scalar equations defining the method applied to (3.39) can be arranged in the form

$$[I_q \otimes (I - \lambda h A)] \hat{\varphi}_k - \lambda h \sum_{j=0}^k [\hat{V}_{kj} \otimes E_p] \hat{\varphi}_j = \gamma_k(\lambda h) \quad (3.96)$$

where the components of $\gamma_k(\lambda h)$ have the form $g(\tau_\nu) + \lambda h w_{\nu, 0} g(0)$, since $\tilde{f}(0) = g(0)$.

Using the result (3.94), we note that

$$[I_q \otimes (I - \lambda h A)^{-1}] [\hat{V}_{kj} \otimes E_p] = \hat{V}_{kj} \otimes (I - \lambda h A)^{-1} E_p = \hat{V}_{kj} \otimes N(\lambda h).$$

Multiplying (3.96) by $I_q \otimes (I - \lambda h A)^{-1}$, yields

$$\hat{\varphi}_k - \lambda h \sum_{j=0}^k [\hat{V}_{kj} \otimes N(\lambda h)] \hat{\varphi}_j = \Gamma_k(\lambda h) \quad (3.97)$$

where $\Gamma_k(\lambda h) = (I_q \otimes (I - \lambda h A)^{-1}) \gamma_k(\lambda h)$.

By virtue of the relations (3.89), (3.90) and (3.93) we find

$$\sum_{\ell=0}^{m+1} A_\ell \hat{V}_{n+1-\ell, j} = \hat{B}_{n+1-j}, \quad (3.98)$$

where $\hat{B}_j = 0$ if $j \notin \{0, 1, \dots, m+1\}$.

It follows that, if we multiply (3.97) by $A_\ell \otimes I$ and sum over ℓ we find

$$\sum_{\ell=0}^{m+1} (A_\ell \otimes I) \hat{\varphi}_{n+1-\ell} - \lambda h \sum_{\ell=0}^{m+1} \hat{B}_\ell \otimes N(\lambda h) \hat{\varphi}_{n+1-\ell} = \sum_{\ell=0}^{m+1} (A_\ell \otimes I) \Gamma_{n+1-\ell}. \quad (3.99)$$

The auxiliary polynomial for this relation, which allows a study of

internal stability, is

$$\det \left[\sum_{\ell=0}^{m+1} (\underline{A}_\ell \otimes \underline{I} - \lambda h \underline{\hat{B}}_\ell \otimes \underline{N}(\lambda h)) \right]_{\mu^{m+1-\ell}}. \quad (3.100)$$

3.6.2 Full step stability

Analysis of (3.100) provides criteria for block-stability, but for an analysis of full-step stability we proceed as follows.

Multiplying (3.99) by $\underline{I}_q \otimes \underline{E}_p$ yields

$$\sum_{\ell=0}^{m+1} (\underline{A}_\ell \otimes \underline{E}_p) \hat{\varphi}_{n+1-\ell} - \lambda h \sum_{\ell=0}^{m+1} \underline{\hat{B}}_\ell \otimes (\underline{E}_p \underline{N}(\lambda h)) \hat{\varphi}_{n+1-\ell} = \sum_{\ell=0}^{m+1} (\underline{A}_\ell \otimes \underline{E}_p) \underline{\Gamma}_{n+1-\ell} \quad (3.101)$$

We observe that $\underline{E}_p = \underline{e} \underline{e}^T$ and hence

$$\begin{aligned} \underline{E}_p \underline{N}(\lambda h) &= \underline{e} \underline{e}^T \underline{d}(\lambda h) \underline{e}^T \\ &= \hat{\mu}(\lambda h) \underline{E}_p \end{aligned}$$

so that (3.101) reduces to

$$\sum_{\ell=0}^{m+1} (\underline{A}_\ell \otimes \underline{E}_p) \hat{\varphi}_{n+1-\ell} - \lambda h \hat{\mu} \sum_{\ell=0}^{m+1} (\underline{\hat{B}}_\ell \otimes \underline{E}_p) \hat{\varphi}_{n+1-\ell} = \sum_{\ell=1}^{m+1} (\underline{A}_\ell \otimes \underline{E}_p) \underline{\Gamma}_{n+1-\ell} \quad (3.102)$$

Let us define

$$\underline{\tilde{\Phi}}_k = [\tilde{f}((kq+1)h), \tilde{f}((kq+2)h), \dots, \tilde{f}((k+1)qh)]^T$$

where $\tilde{f}((kq+r)h)$ is of course $\underline{e}_p^T \varphi_{kq+r}$. Then (3.102) states no more and no less than

$$\sum_{\ell=0}^{m+1} \underline{A}_\ell \underline{\tilde{\Phi}}_{n+1-\ell} - \lambda h \hat{\mu}(\lambda h) \sum_{\ell=0}^{m+1} \underline{\hat{B}}_\ell \underline{\tilde{\Phi}}_{n+1-\ell} = \sum_{\ell=0}^{m+1} \underline{A}_\ell \underline{\gamma}_{n+1-\ell} \quad (3.103)$$

for appropriate vectors $\underline{\gamma}_\ell$ derived from $\underline{\Gamma}_\ell$. The stability of (3.103) governs the full-step stability of the method.

Example 3.7

Consider the mixed method based on the quadrature weights discussed in Examples 2.4 and 2.14 and the Runge-Kutta tableau

1	0				
1	0				
0	$\frac{1}{2}d_1$	1	0		
0	$\frac{1}{2}d_2$	0	1		
0	$\frac{4}{3}d_1$	0	$\frac{1}{3}d_1$	1	0
0	$\frac{4}{3}d_2$	0	$\frac{1}{3}d_2$	0	1
0	$\frac{4}{3}d_1$	0	$\frac{5}{6}d_1$	0	$\frac{1}{2}d_1$
0	$\frac{4}{3}d_2$	0	$\frac{5}{6}d_2$	0	$\frac{1}{2}d_2$

and so on and a direct approach via differencing allows a stability analysis. For full-step stability the result may be obtained directly by substitution in the determinantal expression given in Theorem 3.10. From Example 2.14 we know that we may take $m = 1$,

$$\underline{\hat{A}}_1 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \underline{\hat{A}}_0 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad \underline{\hat{B}}_1 = \begin{bmatrix} \frac{1}{3} & \frac{4}{3} \\ 0 & 0 \end{bmatrix}, \quad \underline{\hat{B}}_0 = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

which yields $\underline{\hat{A}}_2 = 0$, $\underline{\hat{A}}_1 = \underline{\hat{A}}_1$, $\underline{\hat{A}}_0 = \underline{\hat{A}}_0$ and

$$\underline{\hat{B}}_2 = \begin{bmatrix} 0 & \frac{1}{3} \\ 0 & 0 \end{bmatrix}, \quad \underline{\hat{B}}_1 = \begin{bmatrix} \frac{4}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} \end{bmatrix}, \quad \underline{\hat{B}}_0 = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}.$$

From Example 3.2 we know that $\hat{\mu} \equiv \hat{\mu}(\lambda h) = (1 + \frac{1}{2}\lambda h)/(1 - \frac{1}{2}\lambda h)$.

By Theorem 3.10 we require the characteristic polynomial

$$\begin{vmatrix} \mu^2 - \mu(1 + \frac{4}{3}\lambda h \hat{\mu}) & -\frac{1}{3}\lambda h \hat{\mu} \mu - \frac{1}{3}\lambda h \hat{\mu} \\ -(1 + \frac{1}{2}\lambda h \hat{\mu})\mu^2 & \mu^2 - \frac{1}{2}\lambda h \hat{\mu} \mu \end{vmatrix}$$

$$= \mu^2 \{ \mu^2 - (1 + \frac{13}{6} \lambda h \hat{\mu} + \frac{1}{6} \lambda^2 h^2 \hat{\mu}^2) \mu + \frac{1}{6} \lambda h \hat{\mu} + \frac{1}{2} \lambda^2 h^2 \hat{\mu}^2 \}$$

to be a von Neumann polynomial. Denoting its zeros by $\mu_1, \mu_2, \mu_3, \mu_4$ we find $\mu_1 = \mu_2 = 0$, $\mu_3 \approx \frac{1}{6} \lambda h \hat{\mu} + \frac{1}{6} \lambda^2 h^2 \hat{\mu}^2$, $\mu_4 \approx 1 + 2 \lambda h \hat{\mu}$ (and, indeed we find $\mu_4 = \exp(2 \lambda h) + O(h^3)$ as might be expected), on substituting the value of $\hat{\mu}$. If $\mu_3 = \mu_4$ we must check for semi-simplicity.

We may remark that the block-reducible method considered here is actually equivalent to a two-cyclic linear multistep method.

3.7 Extensions

The preceding analysis based on the structure of Runge-Kutta methods, has resulted in recurrence relations and stability polynomials which permit stability results for the "basic" test equation (3.3). Here, we indicate possible extensions to the analysis.

Our discussion has entailed the derivation of recurrence relations (3.29) of the form $\tilde{\Phi}_{k+1} = \tilde{M} \tilde{\Phi}_k + \tilde{\gamma}_{k+1}$, where \tilde{M} depends on λh and is independent of k and the components of $\tilde{\Phi}_k$ are the values of the approximate solutions. A stability condition, resulting in definitions of "absolute" stability, is the requirement that $\rho(\tilde{M}) < 1$, where $\rho(\tilde{M})$ denotes the spectral radius, and that if $\rho(\tilde{M}) = 1$ then \tilde{M} must be of class M . Alternatively, we may ask that $\|\tilde{M}(\lambda h)\| < 1$ in some preselected norm. The remarks of Stetter [25, Section 3.5.5] can be paralleled here, and one may seek regions in the λh -plane, wherein $\rho(\tilde{M}) < \rho'$, $\rho' < 1$ for $\tilde{M} \equiv \tilde{M}(\lambda h)$. Given the nature of the vectors $\tilde{\Phi}_k$ in (3.29) and the behaviour of the solution of the test equation it is possible to define relative stability concepts (Baker [4]) which can also be analysed in terms of the eigenvalues of $\tilde{M} = \tilde{M}(\lambda h)$.

The analysis presented here provides a necessary foundation on which to develop a theory for more involved test equations. With regard

to the practical conclusions which may be drawn from the work, it is appropriate to issue a caveat. We recall that Baker [4, p.763] wrote as follows: "It is, of course, the very simplicity of the (test) equation which makes it ideal for mathematical analysis, and a study of this simple case does give some genuine insight..... What we must guard against is an unquestioning acceptance that a method which is suitable for this special equation is suitable for more complicated equations." As we observed in §3.2.6, it now seems generally accepted that a necessary (but not sufficient) requirement for an all-purpose method is that the method should be suitable for the test equation (3.3). Thus, the analysis developed here for (3.3) can readily be extended to more general test equations and we note the results in [1], [2], [3] for test equations of the form

$$f(x) - \int_0^x \sum_{r=0}^R \lambda_r (x-y)^r f(y) dy = g(x).$$

For such equations the concepts of stability and asymptotic stability depend on R and are not covered by Definition 3.4. The appropriate definitions result in generalizing the concept of A-stability (applicable to methods for equation (3.3)) to a concept of $(A;R)$ -stability (see [2]), for $R = 1, 2, 3, \dots$. A stable method need not be $(A;R)$ -stable, for $R \geq 1$.

Other directions in which our analysis can be extended arise when we seek to mimic properties other than those represented by Definition 3.4. Thus, for example, we motivated the definitions of stability and asymptotic stability by classifying the response of $f(x)$ to *constant* perturbations in $g(x)$. When considering the effect of more general perturbations it is natural to turn explicitly to the properties of, in the linear case, the resolvent kernel, and (for the

numerical methods) the inverse of the infinite block-lower-triangular matrix of coefficients in the formulae (3.28) defining the method. The differencing procedure by which, in the case of (3.3), the latter equations are reduced to the form (3.29) is in effect an elimination procedure permitting such an approach.

Finally, we may enquire what extensions are possible when (3.3) is replaced by a more general equation with, for example, separable kernels (see [6], [14], [15]). A stability analysis may result in recurrence relations of the form

$$\varphi_{k+1} = \underline{M}_k \varphi_k + \underline{\gamma}_k$$

where \underline{M}_k depends on k and where strict stability requires $\Pi_k \underline{M}_k \neq 0$. The *local* stability criterion $\rho(\underline{M}_k) < 1$, for all k , is necessary but not sufficient (if $\rho(\underline{M}_k) = 1$ then \underline{M}_k is required to be of class M) whilst the *local* block-stability criterion $\|\underline{M}_k\| < 1$, for all k is sufficient but not necessary. When $\underline{M}_k \equiv \underline{M}$, the analysis for (3.3) provides some insight into which tests are meaningful in the more general case, and the limitations of such tests. We may compare, for example, regions of block stability in differing norms for the special case, in order to gain insight for the more general case.

1. AMINI, S. & BAKER, C.T.H. Further stability analysis of numerical methods for Volterra integral equations of the second kind. Numer. Anal. Tech. Rep. No.47, University of Manchester (1980).
2. AMINI, S., BAKER, C.T.H., van der HOUWEN, P.J. & WOLKENFELT, P.H.M. Stability analysis of numerical methods for Volterra integral equations with polynomial kernels. Rep. NW 109/81, Math. Centrum, Amsterdam (1981).
3. AMINI, S., BAKER, C.T.H. & WILKINSON, J.C. Basic stability analysis of Runge-Kutta methods for Volterra integral equations of the second kind. Numer. Anal. Tech. Rep. No.46, University of Manchester (1980).
4. BAKER, C.T.H. The numerical treatment of integral equations. Clarendon Press, Oxford (1977, reprinted 1978).
5. BAKER, C.T.H. Runge-Kutta methods for Volterra integral equations of the second kind. Lecture Notes in Mathematics No. 630 Springer-Verlag (1978) pp.1-13.
6. BAKER, C.T.H. Structure of recurrence in the study of stability in the numerical treatment of Volterra integral and integro-differential equations. J. of Integral Equations 2 (1980) pp.11-39.
7. BAKER, C.T.H. & KEECH, M.S. Stability regions in the numerical treatment of Volterra integral equations. SIAM J. Numerical Analysis 15 (1978) pp.394-417.
8. BAKER, C.T.H. & WILKINSON, J.C. Stability analysis of Runge-Kutta methods applied to a basic Volterra integral equation. J. Austral. Math. Soc. (Series B) 22 (1981) pp.515-538.
9. BOWNS, J.M. & CUSHING, J.M. Some stability theorems for systems of Volterra integral equations. Appl. Anal. 5 (1975) pp.65-77.
10. HAHN, W. Theory and application of Liapunov's direct method. Prentice-Hall, Englewood Cliffs (1963).
11. HAIRER, E. & LUBICH, Ch. On the stability of Volterra-Runge-Kutta methods. Sonderforschungsbereich 123, Universität Heidelberg Preprint No. 194 (1982).

12. HENRICI, P. Discrete variable methods in ordinary differential equations. John Wiley & Sons, Inc., New York & London (1962).
13. van der HOUWEN, P.J. Construction of integration formulas for initial value problems. North-Holland, Amsterdam (1977).
14. van der HOUWEN, P.J. On the numerical solution of Volterra integral equations of the second kind, -I Stability;
- II Runge-Kutta methods (with BLOM, J.G.) Reps. NW42/77 & 61/78, Math Centrum, Amsterdam (1977, 1978).
15. van der HOUWEN, P.J. & WOLKENFELT, P.H.M. On the stability of multi-step formulas for Volterra integral equations of the second kind. Rep. NW 59/78, Math Centrum, Amsterdam (1978).
16. LAMBERT, J.D. Computational methods in ordinary differential equations. Wiley (1973).
17. LINZ, P. Numerical methods for Volterra integral equations with applications to certain boundary value problems. Ph.D. Thesis, University of Wisconsin, Madison (1968).
18. LUBICH, Ch. On the stability of linear multistep methods for Volterra equations. Doctoral Thesis, Univ. Innsbruck (1982).
19. MILLER, K.S. Linear difference equations. Menlo Park (California): Benjamin Inc., (1968).
20. NEVANLINNA, O. Positive quadratures for Volterra equations. Computing 16 (1976) pp.349-357.
21. NEVANLINNA, O. On the numerical solution of some Volterra equations on infinite intervals. Rep. 2 Institut Mittag-Leffler, Djursholm (1976).
22. NOBLE, B. Instability when solving Volterra integral equations of the second kind by multistep methods. Lecture Notes in Mathematics No. 109 Springer-Verlag (1969) pp.23-39.
23. ORTEGA, J.M. Numerical Analysis: a second course. Academic Press, New York (1973).

24. PALEY, R.E.A.C. & WIENER, N. Notes on the theory and application of Fourier transforms, VII-On the Volterra equation. Amer.Math.Soc.Trans., Vol 35 (1933) pp.785-791.
25. STETTER, H. Analysis of discretization methods for ordinary differential equations. Springer, New York (1973).
26. TSALYUK, Z.B. Volterra integral equations. J.Soviet.Math. 12 (1979) pp.715-758.
27. VARGA, R.S. Matrix iterative analysis, Prentice Hall (1962).
28. WOLKENFELT, P.H.M. The numerical analysis of reducible quadrature methods for Volterra integral and integro-differential equations Acad.Proefschrift; Amsterdam (1981).
29. WOLKENFELT, P.H.M., van der HOUWEN, P.J. & BAKER, C.T.H. Analysis of numerical methods for second kind Volterra equations by embedding techniques. J. Int. Eqns. 3 (1981) pp.61-82.

CHAPTER 4

4. Basic stability analysis of Runge-Kutta methods for Volterra
 integro-differential equations.

4.1 Numerical methods and stability

4.1.1 Perspective

4.1.2 Extended Runge-Kutta methods and mixed quadrature -
 Runge-Kutta methods.

4.2 Extended Runge-Kutta methods and stability definitions

4.2.1 Stability of Runge-Kutta methods

4.2.2 Extended Runge-Kutta methods

4.2.3 Some analytic comments

4.3 Runge-Kutta/mixed quadrature - Runge-Kutta methods

4.3.1 Runge-Kutta/mixed methods using reducible rules

4.3.2 Runge-Kutta/mixed methods using block-reducible rules

4.4 Mixed quadrature -RK methods/mixed quadrature -RK methods

4.4.1 Reducible rules.

4.2.2 Block-reducible rules

4.5 A-Stability

4.6 Extensions

4.1 Numerical methods and stability

In the previous chapter and in [5] we discussed classical mixed and extended Runge-Kutta methods applicable to the numerical solution of Volterra integral equations of the second kind. We presented a basic stability analysis of such Runge-Kutta methods for a simple test equation of the form

$$f(x) - \lambda \int_0^x f(y) dy = g(x). \quad (4.1)$$

We develop, here, a similar class of Runge-Kutta methods for the numerical solution of the Volterra integro-differential equation introduced in §1.5

$$f'(x) = G(x, f(x), \int_0^x H(x, y, f(y)) dy) + d(x) \quad (x \geq 0). \quad (4.2)$$

Background reading is provided by Baker [2], Feldstein and Sopka [9], Linz [13], Makroglou, [14], McKee [16], Mocarsky [17] and Tavernini [18], etc.

We shall investigate stability results for this particular class of methods applied to the "basic" integro-differential equation

$$f'(x) = \xi f(x) + \eta \int_0^x f(t) dt + d(x). \quad (4.3)$$

To explain our interest in this latter equation and for additional work relating to the stability of (4.3) we cite the more general analysis of Baker, Makroglou and Short [3]. The study of (4.3) may be regarded as preliminary step towards the study of more general test equations. (Stability of various methods is also considered in [1], [5], [8], [12] and [19].)

Our interest is directed primarily towards stability of Runge-Kutta methods of classical and modified form. Those of classical form are

investigated in this chapter. The γ -modified methods and a search for a unified analysis of stability polynomials are pursued in chapter 5.

In §4.2 we re-examine those stability definitions introduced in §3.4 in connection with Runge-Kutta methods for the second kind Volterra equation and consider their relevance to the study of stability of Runge-Kutta methods applied to (4.3).

In §4.3 and §4.4 we introduce an analysis of the R-K/mixed quadrature -R-K methods with reducible and block-reducible rules and present results which are an extension of those of Chapter 3 and [5] to the treatment of integro-differential equations. We discuss the mixed quadrature -R-K/mixed quadrature -R-K methods introduced in [1] but which receive greater attention in [6].

In §4.5 we enlarge on some comments made in [5] considering the A-stability of an overall method when the associated quadrature and extended R-K methods are A-stable.

4.1.1 Perspective

Equation (4.2) provides an example of a functional differential equation. Included as a special case is the initial-value problem for an ordinary differential equation:

$$f'(x) = F(x, f(x)) \quad (f(0) = f_0, x \geq 0). \quad (4.4)$$

Runge-Kutta methods for this equation are well established and are defined in §2.3.1. For convenience, we recall that, in terms of the Runge-Kutta tableau $[\theta|A]$ of (2.33), the Runge-Kutta method may be defined by the formulae

$$f_{i,r} = f_{i-1,p} + h \sum_s A_{rs} F(ih + \theta_s h, f_{i,s}), \quad (r = 0, 1, \dots, p) \quad (4.5)$$

$$f_{i-1,p} = \tilde{f}_i \approx f(ih)$$

in which we take $f_{i,r}$ to be an approximation to $f(ih+\theta_r h)$ and $f_{-1,p} = f_0$. The equations (4.5) may be regarded as discretizations of the integrated form of (4.4):

$$f(ih+\theta_r h) - f(ih) = \int_{ih}^{ih+\theta_r h} F(y, f(y)) dy \quad (r = 0, 1, \dots, p) \quad (4.6)$$

employing quadrature rules, using abscissae $\theta_0, \theta_1, \dots, \theta_p$ and weights $A_{r0}, A_{r1}, \dots, A_{rp}$ to approximate $\int_0^{\theta_r h} \varphi(y) dy$:

$$\int_0^{\theta_r h} \varphi(y) dy \cong \sum_{s=0}^p A_{rs} \varphi(\theta_s h), \quad (r = 0, 1, \dots, p) \quad (4.7)$$

When considering (4.2), a process similar to that above yields

$$f(ih+\theta_r h) - f(ih) = \int_{ih}^{ih+\theta_r h} G(y, f(y), z(y)) dy + d(ih+\theta_r h) - d(ih) \quad (4.8)$$

where

$$z(y) = \int_0^y H(y, t, f(t)) dt, \quad (4.9)$$

and discretizing as in (4.5) provides equations

$$f_{i,r} = f_{i-1,p} + h \sum_s A_{rs} G(ih+\theta_s h, f_{i,s}, z_{i,s}) + d(ih+\theta_r h) - d(ih), \quad (r = 0, 1, \dots, p) \quad (4.10)$$

$$f_{i-1,p} = \tilde{f}_i \approx f(ih)$$

which require approximations $z_{i,s}$ to the values of $z(y)$ in (4.9) at $(ih+\theta_s h)$. To proceed, we require computable approximations $z_{i,s}$ and for this purpose we turn to techniques common in the treatment of Volterra integral equations. In order to discretize (4.9) it has been conventional, when studying a wide class of methods, to re-index the variables (cf. §2.3.3) and we write

$$\tilde{f}_j = f(\tau_j) = f_{i,r} \text{ where } \tau_j = ih + \theta_r h, r = 0, 1, \dots, p \quad (4.11)$$

successively for $i = 0, 1, 2, \dots$ with $j = i(p+1) + r + 1$, $\tau_0 = 0$ and $\tilde{f}_0 = f_0$. Denoting the approximation to $z(\tau_j)$ by \tilde{z}_j we see that the problem of discretizing (4.9) then reduces to the choice of values Ω_{jk} in approximations of the form (see (2.52), (2.53), (2.54) and (2.55))

$$\tilde{z}_j = h \sum_k \Omega_{jk} H(\tau_j, \tau_k, f(\tau_k)). \quad (4.12)$$

We shall return to the construction of numerical methods entirely from the viewpoint of integral equations in §4.1.2. By this means we can draw upon our earlier results in chapter 3. Here we turn to address the question of stability.

In previous chapters we have observed that stability studies are concerned with the reaction of systems to perturbations. In numerical analysis, one considers the response of a numerical scheme to perturbations (for example perturbations in the given initial value for an initial-value problem). Often, one hopes that this response will model that in the analytical problem. The general equation is too difficult to analyze and requires a choice of test equation.

In the early studies the equation

$$f'(x) = \lambda f(x) \quad (4.13)$$

was taken in the study of (4.4) but more general equations have now been considered. Whilst choice of (4.13) as a test equation provides genuine practical insight, subsequent work has highlighted what might be regarded as self-evident: that any choice of test equation has inherent limitations. Equation (4.3) is a simple extension of (4.13).

Equation (4.3) was first employed as a test equation by Brunner and Lambert [8]. The usefulness of (4.3) as a test equation for studying (4.2) has been demonstrated earlier [3, 11, 15] by example, but our previous remarks concerning (4.13) apply a fortiori to (4.3).

There is a sense in which determining the limitations of a test equation requires that the stability analysis should first be performed. Early work on the stability of methods for (4.2) has concerned generalizations of linear multi-step methods, with the exception of the work of Baker, Makroglou and Short [3] where Runge-Kutta methods are included in the formalism. Our present purpose is to provide precise stability polynomials for various Runge-Kutta methods. By adopting the formation which we developed earlier in chapter 3 for integral equations the derivation of the stability polynomials is elegant by its straight forwardness.

The stability polynomials are a tool in the study of "stability regions" as well as relative stability and other features not pursued here. Whilst stability regions may be best investigated computationally the theoretical approach provides interesting challenges and is required to establish A-stability (see below) and related results.

4.1.2. Extended Runge-Kutta methods and mixed quadrature-Runge-Kutta methods.

We may write (4.2) in the form

$$f(x) = \int_0^x G(s, f(s), z(s)) ds + d(x) \quad (4.14a)$$

$$z(x) = \int_0^x H(x, t, f(t)) dt. \quad (4.14b)$$

We have seen in §4.1.1 how a R-K method for (4.2) may be derived, but such methods also follow on applying the ideas of [5] to the coupled integral equations (4.14). We require the Runge-Kutta parameters $\theta_r, A_{rs} (r, s = 0, 1, \dots, p)$, $\theta_p = 1$ [5, Eq. (1.10)], defining the tableau $[\tilde{\theta} | \tilde{A}]$.

For the extended Runge-Kutta method we set $x = ih + \theta_r h$ in (4.14a)

and discretize using the rules employed in extended R-K methods for integral equations. We then have

$$\begin{aligned} \tilde{f}_{i(p+1)+r+1} = & h \sum_{k=0}^{i-1} \sum_{s=0}^p A_{ps} G(kh+\theta_s h, \tilde{f}_{k(p+1)+s+1}, \tilde{z}_{k(p+1)+s+1}) \\ & + h \sum_{s=0}^p A_{rs} G(ih+\theta_s h, \tilde{f}_{i(p+1)+s+1}, \tilde{z}_{i(p+1)+s+1}) + d(ih+\theta_r h). \end{aligned} \quad (4.15)$$

Here, $\tilde{f}_{k(p+1)+s+1} \approx f(kh+\theta_s h)$, $\tilde{z}_{k(p+1)+s+1} \approx z(kh+\theta_s h)$.

Equivalently, we obtain, on differencing (4.15), the equations (which are more suitable for computational use)

$$\begin{aligned} \tilde{f}_{i(p+1)+r+1} - \tilde{f}(ih) = & h \sum_{s=0}^p A_{rs} G(ih+\theta_s h, \tilde{f}_{i(p+1)+s+1}, \tilde{z}_{i(p+1)+s+1}) + \\ & + d(ih+\theta_r h) - d(ih) \end{aligned} \quad (4.16)$$

where

$$\tilde{f}(ih) = \tilde{f}_{i(p+1)}, \quad \tilde{z}(ih) = \tilde{z}_{i(p+1)};$$

we have $\theta_p = 1$. These equations represent (4.10), already obtained.

In the present method we set, in (4.15) or (4.16),

$$\begin{aligned} \tilde{z}_{i(p+1)+r+1} = & h \sum_{k=0}^{i-1} \sum_{s=0}^p A_{ps} H(ih+\theta_r h, kh+\theta_s h, \tilde{f}_{k(p+1)+s+1}) + \\ & + h \sum_{s=0}^p A_{rs} H(ih+\theta_r h, ih+\theta_s h, \tilde{f}_{i(p+1)+s+1}). \end{aligned} \quad (4.17)$$

The combination (4.16), (4.17) will be discussed in §4.2.2. For the mixed quadrature-Runge-Kutta methods to be considered in §§4.3 and 4.4 we require quadrature rules

$$\int_0^{ih} \varphi(t) dt = h \sum_{k=0}^i \omega_{ik} \varphi(kh) \quad (i=1, 2, \dots; h > 0). \quad (4.18)$$

The mixed quadrature-Runge-Kutta methods briefly discussed in §4.4 then assume the form

$$\begin{aligned} \tilde{f}_{i(p+1)+r+1} = h \sum_{k=0}^i \omega_{ik} G(kh, \tilde{f}(kh), \tilde{z}(kh)) + \\ + h \sum_{s=0}^p A_{rs} G(ih+\theta_s h, \tilde{f}_{i(p+1)+s+1}, \tilde{z}_{i(p+1)+s+1} + d(ih+\theta_r h) \end{aligned} \quad (4.19)$$

where, in this method,

$$\begin{aligned} \tilde{z}_{i(p+1)+r+1} = h \sum_{k=0}^i \omega_{ik} H(ih+\theta_r h, kh, \tilde{f}(kh)) + \\ + h \sum_{s=0}^p A_{rs} H(ih+\theta_r h, ih+\theta_s h, \tilde{f}_{i(p+1)+s+1}). \end{aligned} \quad (4.20)$$

Combining (4.16) with (4.20) provides an attractive intermediate course, and yields the RK/mixed quadrature-RK methods which we analyze in §4.3.

We shall assume reducibility properties (see §2.3.5) of the rules (4.18). Briefly, we suppose that the array of weights (ω_{ik}) can be partitioned into submatrices $\tilde{V}_{n,\ell}$ of order q such that for fixed $\{\tilde{A}_\ell, \tilde{B}_\ell\}$ and for some n_0

$$\sum_{\ell=0}^m \tilde{A}_\ell \tilde{V}_{n-\ell,j} = \tilde{B}_{n-j}, \quad n \geq n_0$$

where $\tilde{A}_\nu = \tilde{B}_\nu = 0$ for $\nu \notin \{0, 1, 2, \dots, m\}$.

Such rules are termed block-reducible (§2.3.5). When $q = 1$ we set $\tilde{A}_\ell = \alpha_\ell$, $\tilde{B}_\ell = \beta_\ell$ and the rules are called (ρ^*, σ^*) -reducible where

$$\rho^*(\mu) = \sum_{\ell=0}^m \alpha_\ell \mu^{m-\ell} \quad \text{and} \quad \sigma^*(\mu) = \sum_{\ell=0}^m \beta_\ell \mu^{m-\ell}.$$

Reducibility properties permit the simplification of (4.19) for practical use so that there is no need to store all previously computed values of $\{\tilde{z}(kh)\}$.

We may write the basic test equation (4.3) in the form

$$f(x) = \xi \int_0^x f(t) dt + \eta \int_0^x z(t) dt + \int_0^x d(\ell) d\ell. \quad (4.21)$$

where

$$z(x) = \int_0^x f(t) dt. \quad (4.22)$$

The methods yield approximations $\tilde{f}_j \equiv \tilde{f}_{i(p+1)+r+1}$ to $f(\tau_j)$ where $\tau_j = ih + \theta_r h$ and $j = i(p+1) + r + 1$ ($i=0,1,2,\dots; r=0,1,\dots,p$), via auxiliary approximations $\tilde{z}_j \equiv \tilde{z}_{i(p+1)+r+1}$ to $z(\tau_j)$.

Associated with $\{\tilde{f}_j\}$ and $\{\tilde{z}_j\}$ we define, for $k = 0, 1, 2, \dots$,

$$\varphi_{k+1} = [\tilde{f}_{k(p+1)+1}, \tilde{f}_{k(p+1)+2}, \dots, \tilde{f}_{(k+1)(p+1)}]^T \quad (4.23)$$

$$\psi_{k+1} = [\tilde{z}_{k(p+1)+1}, \tilde{z}_{k(p+1)+2}, \dots, \tilde{z}_{(k+1)(p+1)}]^T \quad (4.24)$$

For the mixed quadrature-Runge-Kutta methods we also use

$$\varphi_0 = f(0)[1, 1, \dots, 1]^T, \quad \psi_0 = 0.$$

4.2 Extended Runge-Kutta methods and stability definitions

4.2.1 Stability of Runge-Kutta methods

We shall seek recurrence relations between vectors such as φ_k or $[\varphi_k^T, \psi_k^T]^T$ in order to discuss stability. The following definition occurs in §3.4.3, but now $X_\ell \equiv X_\ell(\xi h, \eta h^2)$ since we consider the stability of methods applied to (4.3) throughout.

Definition 4.1

The recurrence

$$\sum_{\ell=0}^m X_\ell \varphi_{n+1-\ell} = \gamma_{n+1}^*, \quad \det[X_0] \neq 0 \quad (4.25)$$

is *stable* if and only if the "auxiliary" or "stability" polynomial

$$\det \left[\sum_{\ell=0}^m X_\ell \mu^{m-\ell} \right]$$

has its zeros on the unit disk centered on the origin, those on the boundary being semi-simple (§3.4.3). (The recurrence is *strictly stable* if it is stable but there are no zeros on the boundary).

For related definitions see §3.4.3. We shall distinguish the following types of stability, which relate to the types of recurrence which we shall obtain.

Definition 4.2

(a) If the vectors $\tilde{\chi}_k$ of Definition 4.1 coincide with the vectors $\tilde{\varphi}_k$ of (4.23) then stability is termed internal block stability (without auxiliary variables). (b) If $\tilde{\chi}_k = [\tilde{\varphi}_k^T, \tilde{\psi}_k^T]^T$ then stability is termed internal block stability with auxiliary variables. (c) If the components of $\tilde{\chi}_k$ are values $\tilde{f}(k_0h), \tilde{f}(k_1h), \dots, \tilde{f}(k_qh)$, where $k_r (r=0,1,\dots,q)$ depend on k , the stability is called full-step stability (without auxiliary variables); full-step stability with auxiliary variables is similarly defined.

The values $\tilde{f}(kh)$, $k = 0,1,2,\dots$ are of primary concern, whilst the values $\tilde{f}(kh+\theta_r h)$ ($r = 0,1,\dots,p-1$) are frequently of secondary interest.

We recall from (3.49) the notation

$$\tilde{N}(\lambda h) = (\tilde{I} - \lambda h \tilde{A})^{-1} \tilde{e} \tilde{e}^T \quad (4.26)$$

where $\tilde{e}_0, \tilde{e}_1, \dots, \tilde{e}_p$ are the successive columns of the identity matrix of order $p+1$ and $\tilde{e} = \tilde{e}_0 + \tilde{e}_1 + \dots + \tilde{e}_p$. $\tilde{N}(\lambda h)$ has p zero eigenvalues and non-trivial eigenvalue

$$\hat{\mu}(\lambda h) = \tilde{e}_p^T (\tilde{I} - \lambda h \tilde{A})^{-1} \tilde{e}_p \quad (4.27)$$

4.2.2 Extended Runge-Kutta methods

We consider the extended method, defined by equation (4.15) or (4.16) with (4.17).

We shall show the relationship between all four types of stability in Definition 4.2 and the criteria $|\hat{\mu}(\lambda_0 h)| \leq 1$, $|\hat{\mu}(\lambda_1 h)| \leq 1$ where λ_0, λ_1 are the roots of $\lambda^2 - \xi\lambda - \eta = 0$.

For the basic equation (4.3), the equations defining the method may be written conveniently in vector form. Equation (4.15) becomes

$$[I - \xi h A] \varphi_{i+1} - \xi h E_p A \sum_{k=0}^i \varphi_k - \eta h A \psi_{i+1} - \eta h E_p A \sum_{k=0}^i \psi_k = \delta_{i+1} \quad (4.28)$$

where $A = [A_{rs}]$, $E_p = e_p e_p^T$ and δ_{i+1} is the vector with components $d(ih + \theta_r h)$. Then equations (4.16) and (4.17) become respectively

$$(I - \xi h A) \varphi_{i+1} - E_p \varphi_i - \eta h A \psi_{i+1} = \delta_{i+1} - E_p \delta_i \quad (4.29)$$

and

$$\psi_{i+1} = h A \varphi_{i+1} + h E_p A \sum_{k=1}^i \varphi_k. \quad (4.30)$$

From (4.30) we deduce that

$$\psi_{i+1} - E_p \psi_i = h A \varphi_{i+1}, \quad (4.31)$$

on replacing i by $i-1$ in (4.30) and multiplying by E_p .

Theorem 4.1 (full-step stability with auxiliary variables)

Let

$$\hat{\mu}(\lambda h) = e_p^T [I - \lambda h A]^{-1} e_p, \quad (4.32)$$

and write

$$\lambda_{0,1} = \frac{1}{2} \left[\xi \pm \sqrt{\xi^2 + 4\eta} \right]. \quad (4.33)$$

Then

$$\tilde{f}((i+1)h) - \lambda_1 \tilde{z}((i+1)h) - \hat{\mu}(\lambda_0 h) [\tilde{f}(ih) - \lambda_1 \tilde{z}(ih)] = \gamma_{i+1}^{(0)} \quad (4.34)$$

for certain values $\gamma_{i+1}^{(0)}$, and for full-step stability with auxiliary variables we require, when $\lambda_0 \neq \lambda_1$, that

$$|\hat{\mu}(\lambda_0 h)| < 1, \quad |\hat{\mu}(\lambda_1 h)| < 1. \quad (4.35)$$

Proof. From (4.33), $\xi = \lambda_0 + \lambda_1$ and $\eta = -\lambda_0\lambda_1$, whilst (4.23) and (4.24) provide $\tilde{f}(ih) = e_p^T \varphi_i$, and $\tilde{z}(ih) = e_p^T \psi_i$ ($i=1,2,\dots$). Now equation (4.29) in terms of λ_0 and λ_1 yields

$$(\tilde{I} - \lambda_0 h A) \varphi_{i+1} - \lambda_1 h A \varphi_{i+1} - E_p \varphi_i + \lambda_0 \lambda_1 h A \psi_{i+1} = \delta_{i+1} - E_p \delta_i.$$

Employing (4.31) we deduce that

$$\begin{aligned} (\tilde{I} - \lambda_0 h A) \varphi_{i+1} - \lambda_1 (\psi_{i+1} - E_p \psi_i) - E_p \varphi_i + \\ + \lambda_0 \lambda_1 h A \psi_{i+1} = \delta_{i+1} - E_p \delta_i. \end{aligned} \quad (4.36)$$

Assuming $\det(\tilde{I} - \lambda_0 h A) \neq 0$,

$$\begin{aligned} (\varphi_{i+1} - \lambda_1 \psi_{i+1}) - (\tilde{I} - \lambda_0 h A)^{-1} E_p (\varphi_i - \lambda_1 \psi_i) \\ = (\tilde{I} - \lambda_0 h A)^{-1} (\delta_{i+1} - E_p \delta_i), \end{aligned} \quad (4.37)$$

and taking the inner-product of (4.37) with e_p^T yields, on recalling (4.32), the result (4.34). Equally, the roles of λ_0, λ_1 being inter-changeable,

$$\begin{aligned} \tilde{f}((i+1)h) - \lambda_0 \tilde{z}((i+1)h) - \hat{\mu}(\lambda_1 h) [\tilde{f}(ih) - \lambda_0 \tilde{z}(ih)] \\ (4.38) \end{aligned}$$

$$= \gamma_{i+1}^{(1)} = e_p^T (\tilde{I} - \lambda_1 h A)^{-1} (\delta_{i+1} - E_p \delta_i).$$

Thus, (4.34) and (4.38) yield

$$\begin{aligned} \begin{bmatrix} \tilde{f}((i+1)h) \\ \tilde{z}((i+1)h) \end{bmatrix} = \begin{bmatrix} 1 & -\lambda_1 \\ 1 & -\lambda_0 \end{bmatrix}^{-1} \begin{bmatrix} \hat{\mu}(\lambda_0 h) & 0 \\ 0 & \hat{\mu}(\lambda_1 h) \end{bmatrix} \begin{bmatrix} 1 & -\lambda_1 \\ 1 & -\lambda_0 \end{bmatrix} \begin{bmatrix} \tilde{f}(ih) \\ \tilde{z}(ih) \end{bmatrix} + \begin{bmatrix} \gamma_{i+1}^{(0)} \\ \gamma_{i+1}^{(1)} \end{bmatrix}, \\ (4.39) \end{aligned}$$

provided that $\lambda_0 \neq \lambda_1$. The recurrence (4.39) is of the form (3.52) with an amplification matrix having eigenvalues $\hat{\mu}(\lambda_{0,1}h)$; thus, (4.35) follows when $\lambda_0 \neq \lambda_1$.

We also have, as a consequence of (4.37), the following result

(see also [4]).

Theorem 4.2 (block-stability with auxiliary variables)

The conditions (4.35) are, for $\lambda_0 \neq \lambda_1$, the conditions for internal stability with auxiliary variables.

Proof

By virtue of (4.37) and the interchangeability of λ_0, λ_1

$$\begin{bmatrix} \tilde{\varphi}_{i+1} \\ \tilde{\psi}_{i+1} \end{bmatrix} = \begin{bmatrix} \tilde{I} - \lambda_1 \tilde{I} \\ \tilde{I} - \lambda_0 \tilde{I} \end{bmatrix}^{-1} \begin{bmatrix} N(\lambda_0 h) & 0 \\ 0 & N(\lambda_1 h) \end{bmatrix} \begin{bmatrix} \tilde{I} - \lambda_1 \tilde{I} \\ \tilde{I} - \lambda_0 \tilde{I} \end{bmatrix} \begin{bmatrix} \tilde{\varphi}_i \\ \tilde{\psi}_i \end{bmatrix} + \begin{bmatrix} \gamma_{i+1}^{(0)} \\ \gamma_{i+1}^{(1)} \end{bmatrix} \quad (4.40)$$

for suitable inhomogeneous terms $\gamma_{i+1}^{(0,1)}$, where $\tilde{E}_p = \tilde{e} \tilde{e}^T$ and where $N(\lambda h) = (\tilde{I} - \lambda h \tilde{A})^{-1} \tilde{E}_p$ has non-trivial eigenvalue $\hat{\mu}(\lambda h)$.

Remark: We here restrict our attention to the case $\lambda_0 \neq \lambda_1$; for the case of equality we refer to [4] with the observation that Property 3.1(b) defined there is satisfied for the extended Runge-Kutta method.

We pause to introduce a lemma which we shall use in the proof of Theorem 4.3.

Lemma 4.1

For suitable scalars Γ_{i+1} ,

$$\tilde{f}((i+1)h) - \nu_0 \tilde{f}(ih) - \eta h \nu_1 \tilde{z}(ih) = \Gamma_{i+1}, \quad (4.41)$$

where

$$\nu_0 \equiv \nu_0(\xi h, \eta h^2) = \tilde{e}_p^T (\tilde{I} - \lambda_0 h \tilde{A})^{-1} (\tilde{I} - \lambda_1 h \tilde{A})^{-1} \tilde{e} \quad (4.42)$$

and

$$\nu_1 \equiv \nu_1(\xi h, \eta h^2) = \tilde{e}_p^T (\tilde{I} - \lambda_0 h \tilde{A})^{-1} (\tilde{I} - \lambda_1 h \tilde{A})^{-1} \tilde{A} \tilde{e}. \quad (4.43)$$

Proof Observe that $\tilde{I} - \xi h \tilde{A} - \eta h^2 \tilde{A}^2 = (\tilde{I} - \lambda_0 h \tilde{A})(\tilde{I} - \lambda_1 h \tilde{A})$ where $\lambda_{0,1}$ are defined by (4.33). From (4.29) and (4.31), elimination of

ψ_{i+1} yields

$$(\underline{I} - \xi h \underline{A} - \eta h^2 \underline{A}^2) \varphi_{i+1} - \underline{E}_p \varphi_i - \eta h \underline{A} \underline{E}_p \psi_i = \underline{\delta}_{i+1} - \underline{E}_p \underline{\delta}_i. \quad (4.44)$$

If $(\underline{I} - \lambda_0 h \underline{A})^{-1} (\underline{I} - \lambda_1 h \underline{A})^{-1}$ exists then applying it to both sides of (4.44) and taking the inner-product of the result with \underline{e}_p^T yields (4.41).

Theorem 4.3 (full step stability without auxiliary variables)

The extended Runge-Kutta method applied to (4.3) displays full step stability (without auxiliary variables) and we have

$$\begin{aligned} \tilde{F}((i+2)h) - [\hat{\mu}(\lambda_1 h) + \hat{\mu}(\lambda_0 h)] \tilde{F}((i+1)h) + \\ + \hat{\mu}(\lambda_0 h) \hat{\mu}(\lambda_1 h) \tilde{F}(ih) = \tau_{i+2} \end{aligned} \quad (4.45)$$

for some scalar τ_{i+2} . This recurrence is stable provided

$$|\hat{\mu}(\lambda_0 h)| \leq 1, \quad |\hat{\mu}(\lambda_1 h)| \leq 1 \quad (4.46)$$

where $\hat{\mu}(\lambda h)$ is defined by (4.27). The stability is strict if and only if the inequalities (4.46) are both strict.

Proof From Lemma 4.1,

$$\tilde{F}((i+2)h) - \nu_0 \tilde{F}((i+1)h) - \eta h \nu_1 \tilde{Z}((i+1)h) = \Gamma_{i+2}; \quad (4.47)$$

subtracting $\hat{\mu}(\lambda_0 h)$ times (4.41) from (4.47) yields

$$\begin{aligned} \tilde{F}((i+2)h) - [\hat{\mu}(\lambda_0 h) + \nu_0] \tilde{F}((i+1)h) + \nu_0 \hat{\mu}(\lambda_0 h) \tilde{F}(ih) \\ + \lambda_0 \lambda_1 h \nu_1 \{ \tilde{Z}((i+1)h) - \hat{\mu}(\lambda_0 h) \tilde{Z}(ih) \} = \Gamma_{i+2} - \hat{\mu}(\lambda_0 h) \Gamma_{i+1}. \end{aligned} \quad (4.48)$$

However, (4.34) yields

$$\begin{aligned} h \nu_1 \lambda_0 \lambda_1 \{ \tilde{Z}((i+1)h) - \hat{\mu}(\lambda_0 h) \tilde{Z}(ih) \} \\ = h \nu_1 \lambda_0 \{ \tilde{F}((i+1)h) - \hat{\mu}(\lambda_0 h) \tilde{F}(ih) - \gamma_{i+1}^{(0)} \} \end{aligned}$$

so that, substituting in (4.48), we obtain

$$\begin{aligned} \tilde{F}((i+2)h) - [\hat{\mu}(\lambda_0 h) + \nu_0 - h \nu_1 \lambda_0] \tilde{F}((i+1)h) \\ + [\nu_0 \hat{\mu}(\lambda_0 h) - h \nu_1 \lambda_0 \hat{\mu}(\lambda_0 h)] \tilde{F}(ih) \\ = \Gamma_{i+2} - \hat{\mu}(\lambda_0 h) \Gamma_{i+1} + h \nu_1 \lambda_0 \gamma_{i+1}^{(0)}. \end{aligned}$$

Writing τ_{i+2} for the latter expression, we deduce (4.45) after elementary manipulation. The "auxiliary" polynomial for (4.45) is $\mu^2 - [\hat{\mu}(\lambda_1 h) + \hat{\mu}(\lambda_0 h)] \mu + \hat{\mu}(\lambda_0 h) \hat{\mu}(\lambda_1 h)$ and (4.46) follows.

Theorem 4.4 (block-stability without auxiliary variables)

The condition for block-stability (without auxiliary variables) is

$$|\hat{\mu}(\lambda_0 h)| \leq 1, \quad |\hat{\mu}(\lambda_1 h)| \leq 1.$$

Proof Eliminate ψ_{i+1} from (4.29) by use of (4.31) to obtain

$$(I - \xi h A - \eta h^2 A^2) \varphi_{i+1} - E_p \varphi_i - \eta h A E_p \psi_i = \delta_{i+1} - E_p \delta_i \quad (4.49)$$

On differencing this equation we obtain

$$(I - \xi h A - \eta h^2 A^2) \varphi_{i+1} - (I - \xi h A - \eta h^2 A^2 + E_p) \varphi_i - \eta h A E_p (\psi_i - \psi_{i-1}) = \gamma_i^\# \quad (4.50)$$

Difference equation (4.30) with i replacing $i-1$, and use the result to eliminate $\psi_i - \psi_{i-1}$ from (4.50). The vectors $\{\varphi_i\}$ satisfy a three-term recurrence

$$(I - \xi h A - \eta h^2 A^2) \varphi_{i+2} - (I - \xi h A - \eta h^2 A^2 + E_p + \eta h^2 A E_p A) \varphi_{i+1} + E_p \varphi_i = \gamma_i^\#,$$

and the associated stability polynomial can be factored as

$$\det \left[\frac{1}{\mu} \{ \mu(I - \lambda_0 h A) - E_p \} (\mu - 1) I + E_p \} \{ \mu(I - \lambda_1 h A) - E_p \} \right]$$

whence the result follows.

Example 4.1

Consider the conventional Runge-Kutta tableau, used in Example 3.2,

$$\begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline \frac{1}{2} & \frac{1}{2} \\ \hline 1 & 1 \end{array} \quad \begin{array}{c} 0 \\ 0 \end{array}$$

where $\hat{\mu}(\lambda h) = (1 + \frac{1}{2} \lambda h)(1 - \frac{1}{2} \lambda h)^{-1}$. The conditions for full-step or block-stability (with or without auxiliary variables) for the

extended Runge-Kutta method applied to (4.3) are $|\hat{\mu}(\lambda_0 h)| \leq 1$ and

$|\hat{\mu}(\lambda, h)| \leq 1$, where $\lambda_{0,1}$ are the roots of $\lambda^2 - \xi \lambda - \eta = 0$. When ξ, η

and h are real this results in $\xi h < 0$ and $\{\xi \pm \sqrt{\xi^2 + 4\eta}\}h < 0$ where $\eta = -\lambda_0\lambda_1$ and $\xi = \lambda_0 + \lambda_1$. Thus, we obtain $\xi h < 0$, and $\eta h^2 \leq 0$ so that the region of stability may be exhibited as in Figure 1.

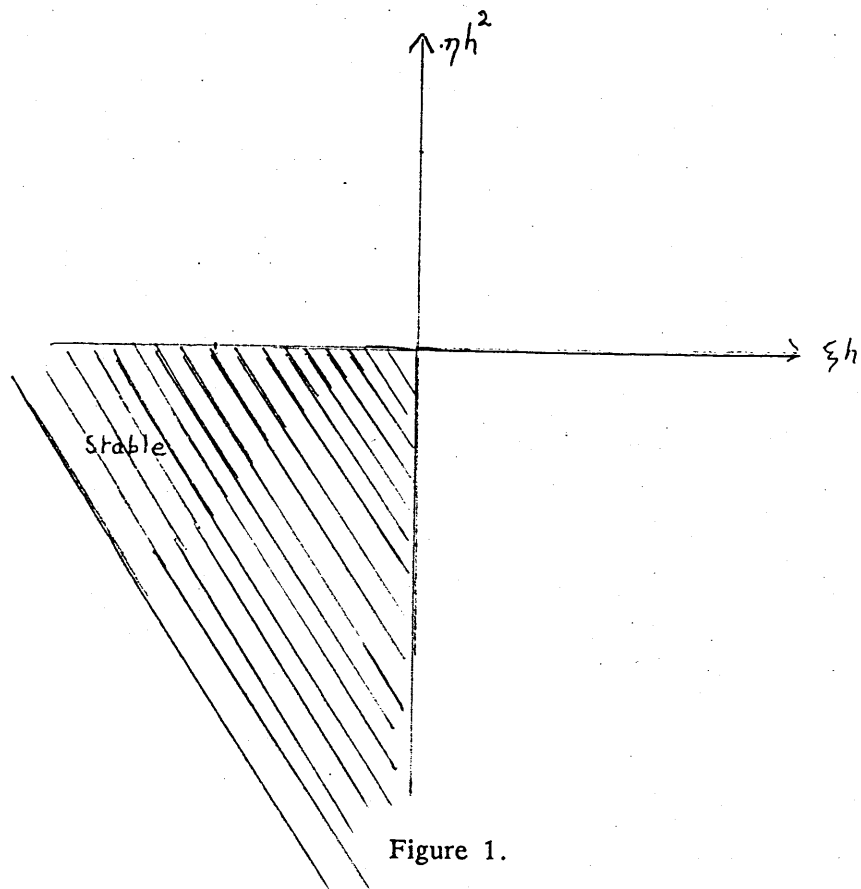


Figure 1.

The region of numerical stability coincides with the region of analytical stability.

4.2.3 Some analytic comments

We pause to comment upon the stability polynomials produced in Section 4.2.2. which require certain functions associated with the Runge-Kutta tableau $[\theta|A]$. The functions of interest are $\hat{\mu}(\lambda h)$, with $\lambda = \lambda_{0,1}$ defined by (4.32) and $\nu_0 = \nu_0(\xi h, \eta h^2)$ and $\nu_1 = \nu_1(\xi h, \eta h^2)$ defined by (4.42) and (4.43). The following results are useful in obtaining specific stability polynomials.

LEMMA 4.2

$$\nu_0 = \hat{\mu}(\lambda_0 h) + \lambda_1 h \nu_1 = \hat{\mu}(\lambda_1 h) + \lambda_0 h \nu_1, \quad (4.51)$$

and

$$\nu_1 = \begin{cases} \hat{\mu}[\lambda_0 h, \lambda_1 h] \equiv \{\hat{\mu}(\lambda_0 h) - \hat{\mu}(\lambda_1 h)\} / \{\lambda_0 h - \lambda_1 h\} & \lambda_0 \neq \lambda_1 \\ \left[\frac{d}{dt} \hat{\mu}(t) \right]_t = \lambda_0 h & \lambda_0 = \lambda_1 \end{cases} \quad (4.52)$$

REMARK

The notation $\hat{\mu}[\lambda_0 h, \lambda_1 h]$ denotes a divided difference. Equations (4.51) and (4.52) permit the expression of ν_0 as a divided difference of the function $\lambda h \hat{\mu}(\lambda h)$. Equation (4.51) was used in the proof of Theorem 4.3.

Conventionally, $\hat{\mu}(\lambda h)$ is a rational approximation to $\exp(\lambda h)$. We have the following result.

Lemma 4.3

if (2.33) is a "conventional" Runge-Kutta array which, for convenience, is reproduced below

$$[\theta|A] = \begin{array}{c|cccc} \theta_0 & A_{00} & \dots & A_{0,p-1} & A_{0,p} \\ \theta_1 & A_{10} & \dots & A_{1,p-1} & A_{1,p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \theta_{p-1} & A_{p-1,0} & \dots & A_{p-1,p-1} & A_{p-1,p} \\ \hline 1 & A_{p0} & \dots & A_{p,p-1} & A_{p,p} \end{array} \quad (4.53)$$

it is usual to take $A_{rp} = 0$ for $r = 0, 1, \dots, p$ and to denote $A_{p,r}$ ($r = 0, 1, \dots, p-1$) by ω_r . Thus, a conventional Runge-Kutta array corresponds to a choice of (4.53) with

$$\tilde{A} = \left[\begin{array}{c|c} \tilde{A}^\# & \tilde{0} \\ \hline \tilde{\omega}^T & 0 \end{array} \right] \quad (4.54)$$

Then,

$$(a) \quad \hat{\mu}(\lambda h) = 1 - \lambda h \tilde{\omega}^T (\tilde{I} - \lambda h \tilde{A}^\#)^{-1} \tilde{1} \tilde{1} \quad (4.55)$$

$$(b) \quad \nu_1(\xi h, \eta h^2) = \tilde{\omega}^T (\tilde{I} - \lambda_0 h \tilde{A}^\#)^{-1} (\tilde{I} - \lambda_1 h \tilde{A}^\#)^{-1} \tilde{1} \tilde{1} \quad (4.56)$$

where $\lambda_{0,1}$ are defined by (4.33) and $\tilde{1} = [1, 1, \dots, 1]^T \in \mathbb{R}^p$ and \tilde{I} denotes the identity matrix of order p .

Remark Lemmas 4.2 and 4.3 permit expressions for ν_0, ν_1 in terms of ξh , ηh^2 and $\tilde{A}^\#, \tilde{\omega}$.

4.3 Runge-Kutta/Mixed quadrature-Runge-Kutta methods

For the Runge-Kutta/mixed quadrature Runge-Kutta methods the results are less transparent. For the basic test equation (4.3) the vector equations associated with the method defined by (4.16) and (4.20) are

$$(\tilde{I} - \xi h \tilde{A}) \tilde{\varphi}_{i+1} - \tilde{E}_p \tilde{\varphi}_i - \eta h \tilde{A} \tilde{\psi}_{i+1} = \tilde{\delta}_{i+1} - \tilde{E}_p \tilde{\delta}_i \quad (i=0, 1, 2, \dots), \quad (4.57)$$

and

$$\tilde{\psi}_{i+1} = h \tilde{A} \tilde{\varphi}_{i+1} + h \sum_{k=0}^i \omega_{ik} \tilde{E}_p \tilde{\varphi}_k, \quad (4.58)$$

where $\tilde{\varphi}_0 = f(0)\tilde{e}$. We shall consider two types of quadrature rules (4.18), those which are reducible (§§2.3.5 and 4.1.2) and those which are block-reducible (ibid).

4.3.1 Runge-Kutta/Mixed using reducible rules

We shall consider quadrature rules (4.18) which, we recall, are

(ρ^*, σ^*) -reducible when

$$\sum_{\ell=0}^m \alpha_{\ell} \omega_{n-\ell, j} = \beta_{n-j}, \quad j = n, n-1, \dots, n-m, \quad (4.59)$$

$$= 0 \quad \text{otherwise}$$

$$\text{and } \rho^*(\mu) = \sum_{\ell=0}^m \alpha_{\ell} \mu^{m-\ell}, \quad \sigma^*(\mu) = \sum_{\ell=0}^m \beta_{\ell} \mu^{m-\ell}. \quad (4.60)$$

We attempt to find stability criteria in terms of ρ^* , σ^* , $\hat{\mu}(\lambda_{0,1}h)$ and $\nu_{0,1}$ defined in (4.42) and (4.43).

From (4.57), eliminating ψ_{i+1} by means of (4.58), we find

$$(\underline{I} - \xi h \underline{A} - \eta h^2 \underline{A}^2) \varphi_{i+1} - \underline{E}_p \varphi_i - \eta h^2 \underline{A} \sum_{k=0}^i \omega_{ik} \underline{E}_p \varphi_k = \underline{\delta}_{i+1} - \underline{E}_p \underline{\delta}_i.$$

Provided that $(\underline{I} - \xi h \underline{A} - \eta h^2 \underline{A}^2)^{-1} = (\underline{I} - \lambda_0 h \underline{A})^{-1} (\underline{I} - \lambda_1 h \underline{A})^{-1}$ exists we obtain

$$\begin{aligned} \varphi_{i+1} &= (\underline{I} - \lambda_0 h \underline{A})^{-1} (\underline{I} - \lambda_1 h \underline{A})^{-1} \underline{E}_p \varphi_i \\ &\quad - \eta h^2 \sum_{k=0}^i \omega_{ik} (\underline{I} - \lambda_0 h \underline{A})^{-1} (\underline{I} - \lambda_1 h \underline{A})^{-1} \underline{A} \underline{E}_p \varphi_k \\ &= (\underline{I} - \lambda_0 h \underline{A})^{-1} (\underline{I} - \lambda_1 h \underline{A})^{-1} (\underline{\delta}_{i+1} - \underline{E}_p \underline{\delta}_i). \end{aligned} \quad (4.61)$$

Employing (4.59) we find, on replacing i by $n-\ell$,

$$\begin{aligned} \sum_{\ell=0}^m \alpha_{\ell} \varphi_{n+1-\ell} &= \{(\underline{I} - \lambda_0 h \underline{A})^{-1} (\underline{I} - \lambda_1 h \underline{A})^{-1} \underline{E}_p\} \sum_{\ell=0}^m \alpha_{\ell} \varphi_{n-\ell} \\ &\quad - \eta h^2 \sum_{\ell=0}^m \beta_{\ell} (\underline{I} - \lambda_0 h \underline{A})^{-1} (\underline{I} - \lambda_1 h \underline{A})^{-1} \underline{A} \underline{E}_p \varphi_{n-2} \\ &= (\underline{I} - \lambda_0 h \underline{A})^{-1} (\underline{I} - \lambda_1 h \underline{A})^{-1} \sum_{\ell=0}^m \alpha_{\ell} (\underline{\delta}_{n+1-\ell} - \underline{E}_p \underline{\delta}_{n-\ell}). \end{aligned} \quad (4.62)$$

We may readily deduce the following result.

Lemma 4.4

A stability polynomial for the RK/mixed quadrature-RK method employing (ρ^*, σ^*) -reducible rules is

$$\mu \rho^*(\mu) - \nu_0 \rho^*(\mu) - \eta h^2 \nu_1 \sigma^*(\mu). \quad (4.63)$$

Proof Apply \underline{e}_p^T to (4.62) and use (4.42) and (4.43) to obtain

$$\sum_{\ell=0}^m \alpha_{\ell} \{f((n+1-\ell)h) - \nu_0 f((n-\ell)h)\} - \eta h^2 \nu_1 \sum_{\ell=0}^m \beta_{\ell} f((n-\ell)h) = \tau_n^* \quad (4.64)$$

(where τ_n^* is the inner product of \tilde{e}_p^T and the right-hand term of (4.62)). The result (4.63) readily follows from (4.64).

Theorem 4.5 (full step stability without auxiliary variables)

$$\text{Let } s(\lambda_0 h, \mu) \equiv \mu \rho^*(\mu) - \lambda_0 h \hat{\mu}(\lambda_0 h) \sigma^*(\mu)$$

and

$$\sigma(\lambda_0 h, \mu) \equiv \rho^*(\mu) - \lambda_0 h \sigma^*(\mu).$$

Then the method of Lemma 4.4 exhibits full-step stability (without auxiliary variables) if and only if

$$s(\lambda_0 h, \mu) - \nu_0 \sigma(\lambda_0 h, \mu) \quad (4.65)$$

is von Neumann, where ν_0 is defined by (4.42).

Proof From the definitions of ν_1 and ν_0 , $\nu_1 = (\nu_0 - \hat{\mu}(\lambda_0 h))/(\lambda_1 h)$ whilst $\eta = -\lambda_0 \lambda_1$. Thus, the auxiliary polynomial (4.63) may be written as (4.65).

Remark

Two families of $\{\rho^*, \sigma^*\}$ -reducible rules are prominent in the literature: the Gregory rules are reducible with $\{\rho^*, \sigma^*\}$ defined by the Adams-Moulton methods; here $\rho^*(\mu) = \mu^m - \mu^{(m-1)}$. The Curtis-Hirschfelder rules are those generated by the Backward-differentiation formulae; here $\sigma^*(\mu) = \mu^m$. The composite versions of the trapezium rule and of the backward Euler rule are the simplest examples of the two families, here $\omega_{ij} = 1$ if $1 \leq j \leq i-1$, $\omega_{i0} = \omega_{ii} = \frac{1}{2}$ and $\omega_{i0} = 0$, $\omega_{ij} = 1$, $1 \leq j \leq i$ respectively.

Example 4.2

Consider the mixed Runge-Kutta method employing the repeated

trapezium rule (Example 2.7) and the Runge-Kutta tableau

$$\begin{array}{c|cc} & 1 & 1 \\ \hline \frac{1}{2} & \frac{1}{2} & 0 \\ \hline 1 & 1 & 0 \end{array}$$

which yields $\hat{\mu}(\lambda h) = (1 + \frac{1}{2} \lambda h)(1 - \frac{1}{2} \lambda h)^{-1}$.

$$\rho^*(\mu) = \mu^{-1}; \quad \sigma^*(\mu) = \frac{1}{2}(\mu+1).$$

$$\nu_0 = \left[1 + \frac{\lambda_0 h}{2} + \frac{\lambda_1 h}{2} - \frac{\lambda_0 \lambda_1 h^2}{4}\right] \left[1 - \frac{\lambda_0 h}{2}\right]^{-1} \left[1 - \frac{\lambda_1 h}{2}\right]^{-1}.$$

$$\nu_1 = \left[1 - \frac{\lambda_0 h}{2}\right]^{-1} \left[1 - \frac{\lambda_1 h}{2}\right]^{-1} \quad \text{and} \quad 1 + \nu_0 = 2\nu_1.$$

$$\begin{aligned} \hat{\mu}(\lambda_0 h) + \hat{\mu}(\lambda_1 h) &= (2 - \frac{1}{2} \lambda_0 \lambda_1 h^2) \left[1 - \frac{\lambda_0 h}{2}\right]^{-1} \left[1 - \frac{\lambda_1 h}{2}\right]^{-1} = (2 - \frac{1}{2} \lambda_0 \lambda_1 h^2) \nu_1 \\ \hat{\mu}(\lambda_0 h) \hat{\mu}(\lambda_1 h) &= \nu_0 + \frac{\lambda_0 \lambda_1}{2} h^2 \nu_1 \end{aligned}$$

The auxiliary polynomial obtained is

$$\begin{aligned} s(\lambda_0 h, \mu) - \nu_0 \sigma(\lambda_0 h, \mu) &= \mu(\mu-1) - \frac{\lambda_0 h}{2} \hat{\mu}(\lambda_0 h)(\mu+1) - \nu_0 [(\mu-1) - \frac{\lambda_0 h}{2} (\mu+1)] \\ &= \mu^2 - \mu \left[1 - \frac{\lambda_1 \lambda_0}{2} h^2 \nu_1 + \nu_0\right] + [\nu_0 + \frac{\lambda_1 \lambda_0}{2} h^2 \nu_1] \\ &= \mu^2 - \mu [\hat{\mu}(\lambda_0 h) + \hat{\mu}(\lambda_1 h)] + \hat{\mu}(\lambda_0 h) \hat{\mu}(\lambda_1 h). \end{aligned}$$

Thus, we have full-step stability (without auxiliary variables) if

$|\hat{\mu}(\lambda_0 h)| \leq 1$ and $|\hat{\mu}(\lambda_1 h)| \leq 1$. (The same result can be deduced for the extended RK method.)

4.3.2 Runge-Kutta/Mixed methods using block-reducible rules

Generalizing the class of quadrature rules leads to added complication. We transfer our attention to the use in mixed methods of quadrature rules (4.18) which are block-reducible (which is to say, those rules which are equivalent to the application of cyclic linear multistep methods for the problem of indefinite integration). We have (c.f. §3.6.1), as a consequence of the block reducibility, the result

$$\sum_{\ell=0}^{m+1} A_{\ell} \hat{V}_{n+1-\ell, j} = \hat{B}_{n+1-j} \quad (4.66)$$

where $\hat{B}_{\ell} = \hat{B}_{\ell-1} J^{\#} + \hat{B}_{\ell} J$, with $J = [\underline{\varepsilon}_1, \underline{\varepsilon}_2, \dots, \underline{\varepsilon}_{q-1}, \underline{0}]$ and $J^{\#} = [\underline{0}, \underline{0}, \dots, \underline{0}, \underline{\varepsilon}_0]$, $\underline{\varepsilon}_0, \underline{\varepsilon}_1, \dots, \underline{\varepsilon}_{q-1}$ being the successive columns of the identity of order q ($\hat{B}_{\ell} = 0$ if $\ell \notin \{0, 1, \dots, m+1\}$). The following result is a natural extension of Theorem 4.5.

Theorem 4.6

Let

$$\hat{\underline{S}}(\lambda_0 h, \mu) \equiv \mu \underline{P}^*(\mu) - \lambda_0 h \mu (\lambda_0 h) \hat{\underline{\Sigma}}^*(\mu)$$

and let

$$\hat{\underline{\Sigma}}(\lambda_0 h, \mu) \equiv \underline{P}^*(\mu) - \lambda_0 h \hat{\underline{\Sigma}}^*(\mu)$$

where $\underline{P}^*(\mu) = \sum_{\ell=0}^m A_{\ell} \mu^{m-\ell}$, $\hat{\underline{\Sigma}}^*(\mu) = \sum_{\ell=0}^{m+1} \hat{B}_{\ell} \mu^{m+1-\ell}$.

Then the RK/mixed quadrature - RK method using block reducible quadrature formulae displays full-step stability without auxiliary variables, if and only if, with the notation (4.33), (4.42).

$$\det(\hat{\underline{S}}(\lambda_0 h, \mu) - \nu_0 \hat{\underline{\Sigma}}(\lambda_0 h, \mu)) \quad (4.67)$$

is von Neumann.

Proof. The vectors $\underline{\varphi}_j, \underline{\psi}_j$, employed earlier, now generate vectors $\hat{\underline{\varphi}}_k, \hat{\underline{\psi}}_k$ on setting

$$\hat{\underline{\varphi}}_{k+1} = [\underline{\varphi}_{kq+1}^T, \dots, \underline{\varphi}_{(k+1)q}^T]^T, \quad k = 0, 1, 2, \dots, \quad (4.68)$$

(and defining $\hat{\underline{\psi}}_k$ similarly) with $\hat{\underline{\varphi}}_0 = f(0) [0, 0, \dots, 0, 1]^T$ in $\mathbb{C}^{q(p+1)}$.

The equations (4.16) and (4.20) defining the method applied to our basic test equation may now be written, using the Kronecker product notation (Definition 3.9) as

$$(\underline{I} \otimes (\underline{I} - \xi h \underline{A})) \hat{\underline{\varphi}}_{i+1} - (\underline{I} \otimes \underline{E}_p) \hat{\underline{\varphi}}_i - \eta h (\underline{I} \otimes \underline{A}) \hat{\underline{\psi}}_{i+1} = \underline{\Delta}_{i+1} \quad (4.69)$$

for a vector $\underline{\Delta}_{i+1}$ depending upon the function d (compare the equivalent formulation (4.57)), and

$$\hat{\underline{\psi}}_{i+1} - h(\underline{I} \otimes \underline{A}) \hat{\underline{\varphi}}_{i+1} - h \sum_{j=0}^{i+1} (\underline{V}_{i+1,j} \otimes \underline{E}_p) \hat{\underline{\varphi}}_j = \underline{0}. \quad (4.70)$$

We may eliminate $\hat{\underline{\psi}}_{i+1}$ between (4.69) and (4.70) on multiplying the latter by $[\underline{I} \otimes \underline{A}] \eta h$, and adding; we obtain

$$\begin{aligned} & (\underline{I} \otimes (\underline{I} - \xi h \underline{A} - \eta h^2 \underline{A}^2)) \hat{\varphi}_{i+1} - (\underline{I} \otimes \underline{E}_p) \hat{\varphi}_i \\ & - \eta h^2 \sum_{j=0}^{i+1} (\underline{V}_{i+1,j} \otimes (\underline{A} \underline{E}_p)) \hat{\varphi}_j = \underline{\Delta}_{i+1} \end{aligned} \quad (4.71)$$

If we now apply $\underline{A}_\ell \otimes \underline{I}$ to (4.71) and sum over ℓ ($\ell = 0, 1, \dots, m$), we obtain

$$\begin{aligned} & \sum_{\ell=0}^m (\underline{A}_\ell \otimes (\underline{I} - \xi h \underline{A} - \eta h^2 \underline{A}^2)) \hat{\varphi}_{n+1-\ell} - \sum_{\ell=0}^m (\underline{A}_\ell \otimes \underline{E}_p) \hat{\varphi}_{n-\ell} \\ & - \eta h^2 \sum_{\ell=0}^{m+1} (\underline{B}_\ell \otimes \underline{A} \underline{E}_p) \hat{\varphi}_{n+1-\ell} = \hat{\Delta}_{n+1} \end{aligned} \quad (4.72)$$

for some suitable term $\hat{\Delta}_{n+1}$.

Let us write

$$\underline{N}_r \equiv \underline{N}_r(\xi h, \eta h^2) = (\underline{I} - \xi h \underline{A} - \eta h^2 \underline{A}^2)^{-1} \underline{A}^r \underline{E}_p \quad (4.73)$$

for $r = 0$ and $r = 1$, where $\underline{E}_p = \underline{e} \underline{e}_p^T$. The matrix \underline{N}_r is of rank unity with non-trivial eigenvalue

$$\underline{e}_p^T (\underline{I} - \xi h \underline{A} - \eta h^2 \underline{A}^2)^{-1} \underline{A}^r \underline{e} = \nu_r(\xi h, \eta h^2), \quad (4.74)$$

the latter being defined for $r = 0$, $r = 1$ by (4.42) and (4.43) respectively.

The notation (4.73) allows a convenient reformulation of (4.72).

Multiplying (4.72) by $\underline{I} \otimes (\underline{I} - \xi h \underline{A} - \eta h^2 \underline{A}^2)^{-1}$ yields

$$\begin{aligned} & \sum_{\ell=0}^m (\underline{A}_\ell \otimes \underline{I}) \hat{\varphi}_{n+1-\ell} - \sum_{\ell=0}^m (\underline{A}_\ell \otimes \underline{N}_0(\xi h, \eta h^2)) \hat{\varphi}_{n-\ell} \\ & - \eta h^2 \sum_{\ell=0}^{m+1} (\underline{B}_\ell \otimes \underline{N}_1(\xi h, \eta h^2)) \hat{\varphi}_{n+1-\ell} = \hat{\gamma}_{n+1} \end{aligned} \quad (4.75)$$

for some vector $\hat{\gamma}_{n+1}$.

The recurrence (4.72) or (4.75) provides the basis for a stability result in terms of internal stability, for full-step stability we require a relation obtained on multiplying (4.75) by $(\underline{I} \otimes \underline{E}_p)$.

Observing that

$$\underline{E}_p \underline{N}_r(\xi h, \eta h^2) = \nu_r(\xi h, \eta h^2) \underline{E}_p, \quad (4.76)$$

we thus deduce from (4.75) that

$$\sum_{\ell=0}^m (\underline{A}_{\ell} \otimes \underline{E}_p) \hat{\varphi}_{n+1-\ell} - \nu_0(\xi h, \eta h^2) \sum_{\ell=0}^m (\underline{A}_{\ell} \otimes \underline{E}_p) \hat{\varphi}_{n-\ell} \quad (4.77)$$

$$-\eta h^2 \nu_1(\xi h, \eta h^2) \sum_{\ell=0}^{m+1} (\underline{B}_{\ell} \otimes \underline{E}_p) \hat{\varphi}_{n+1-\ell} = (\underline{I} \otimes \underline{E}_p) \hat{\gamma}_{n+1}$$

Now consider the vectors $\tilde{\Phi}_k = [\tilde{f}((kq+1)h), \dots, \tilde{f}((k+1)qh)]^T$ which are related to the vectors (4.68) since

$$\tilde{\Phi}_k = [\underline{e}_p^T \varphi_{kq+1}, \underline{e}_p^T \varphi_{kq+2}, \dots, \underline{e}_p^T \varphi_{(k+1)q}]^T. \quad (4.78)$$

Equation (4.77) states that

$$\begin{aligned} \sum_{\ell=0}^m \underline{A}_{\ell} \tilde{\Phi}_{n+1-\ell} - \nu_0(\xi h, \eta h^2) \sum_{\ell=0}^m \underline{A}_{\ell} \tilde{\Phi}_{n-\ell} \\ - \eta h^2 \nu_1(\xi h, \eta h^2) \sum_{\ell=0}^{m+1} \underline{B}_{\ell} \tilde{\Phi}_{n+1-\ell} = \underline{\delta}_{n+1} \end{aligned} \quad (4.79)$$

for a suitably chosen vector $\underline{\delta}_{n+1}$. Expressing ν_1 in terms of ν_0 yields the required result (4.67).

Example 4.3

Consider the mixed -RK method employing block-reducible quadrature formulae of Example 3.7, in which we may take \underline{A} from Example 4.2 and

$$\underline{A}_1 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \underline{A}_0 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \underline{\hat{B}}_2 = \begin{bmatrix} 0 & \frac{1}{3} \\ 0 & 0 \end{bmatrix}, \underline{\hat{B}}_1 = \begin{bmatrix} \frac{4}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} \end{bmatrix}, \underline{\hat{B}}_0 = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}.$$

$$\underline{P}^*(\mu) = \sum_{\ell=0}^1 \underline{A}_{\ell} \mu^{1-\ell} = \underline{A}_0 \mu + \underline{A}_1 = \begin{bmatrix} \mu^{-1} & 0 \\ -\mu & \mu \end{bmatrix}.$$

$$\underline{\Sigma}^*(\mu) = \sum_{\ell=0}^2 \underline{\hat{B}}_{\ell} \mu^{2-\ell} = \underline{\hat{B}}_0 \mu^2 + \underline{\hat{B}}_1 \mu + \underline{\hat{B}}_2 = \begin{bmatrix} \frac{4}{3}\mu & \frac{1}{3}\mu + \frac{1}{3} \\ \frac{1}{2}\mu^2 & \frac{1}{2}\mu \end{bmatrix}.$$

We know that $\hat{\mu} = \hat{\mu}(\lambda h) = (1 + \frac{1}{2}\lambda h)/(1 - \frac{1}{2}\lambda h)$, $\nu_1 = \left[1 - \frac{\lambda_0 h}{2}\right]^{-1} \left[1 - \frac{\lambda_1 h}{2}\right]^{-1}$

and $1 + \nu_0 = 2\nu_1$. If $\hat{\mu}_0 = \mu(\lambda_0 h)$, then by Theorem 4.6 we require

$$\left| \begin{array}{cc} (\mu - \nu_0)(\mu - 1) - \frac{4}{3} \lambda_0 h \mu (\hat{\mu}_0 - \nu_0) & - \frac{\lambda_0 h}{3} (\mu + 1)(\hat{\mu}_0 - \nu_0) \\ -\mu(\mu - \nu_0) - \frac{\lambda_0 h}{2} \mu^2 (\hat{\mu}_0 - \nu_0) & (\mu - \nu_0)\mu - \frac{\lambda_0 h}{2} \mu (\hat{\mu}_0 - \nu_0) \end{array} \right|$$

to be von Neumann. After some manipulation this expression reduces to

$$\begin{aligned} & \mu[\mu^3 - (2\nu_0 + 1 + \frac{13}{6} \eta h^2 \nu_1 + \frac{1}{6} \eta^2 h^4 \nu_1^2) \mu^2 + \\ & + (\nu_0^2 + \frac{13}{6} \eta h^2 \nu_1 \nu_0 + 2\nu_0 + \frac{1}{6} \eta h^2 \nu_1 + \frac{1}{2} \eta^2 h^4 \nu_1^2) \mu - \nu_0(\nu_0 + \frac{1}{6} \eta h^2 \nu_1)] \end{aligned}$$

4.4 Mixed quadrature -RK methods/Mixed quadrature - RK methods

When quadrature rules are reducible or block-reducible the equations (4.18) simplify to the application of a linear multistep or a cyclic linear multistep method to the problem of indefinite integration and there is a corresponding simplification in (4.19).

Since both reducible and block-reducible quadrature rules share common features we shall first treat the unsimplified equation (4.19), which in vector form, for the basic equation (4.3), becomes

$$(\underline{I} - \xi h \underline{A}) \underline{\varphi}_{i+1} - \xi h \sum_{k=0}^i \omega_{ik} \underline{E}_p \underline{\varphi}_k \quad (4.80)$$

$$- \eta h \underline{A} \underline{\psi}_{i+1} - \eta h \sum_{k=0}^i \omega_{ik} \underline{E}_p \underline{\psi}_k = \underline{\delta}_{i+1},$$

in the notation (4.23), (4.24). Likewise, (4.20) becomes

$$\underline{\psi}_{i+1} = h \underline{A} \underline{\varphi}_{i+1} + h \sum_{k=0}^i \omega_{ik} \underline{E}_p \underline{\varphi}_k. \quad (4.81)$$

Substituting (4.81) in (4.80) and collecting terms yields

$$\begin{aligned} & (\underline{I} - \xi h \underline{A} - \eta h^2 \underline{A}^2) \underline{\varphi}_{i+1} \\ & - \{(\xi h \underline{I} + \eta h^2 \underline{A}) \underline{E}_p + \eta h^2 \underline{E}_p \underline{A}\} \sum_{k=0}^i \omega_{ik} \underline{\varphi}_k = \end{aligned} \quad (4.82)$$

$$- \eta h^2 \tilde{E}_p \sum_{k=0}^i \omega_{ik} \sum_{j=0}^{k-1} \omega_{k-1,j} \varphi_j = \delta_{i+1}.$$

4.4.1 Reducible Rules

We shall now assume that the rules (4.18) are $\{\rho^*, \sigma^*\}$ -reducible. Then, from (4.82),

$$\begin{aligned} & (I - \xi h \tilde{A} - \eta h^2 \tilde{A}^2) \sum_{\ell_1=0}^m \alpha_{\ell_1} \varphi_{n+1-\ell_1} \\ & - ((\xi h I + \eta h^2 \tilde{A}) \tilde{E}_p + \eta h^2 \tilde{E}_p \tilde{A}) \sum_{\ell_1=0}^m \beta_{\ell_1} \varphi_{n-\ell_1} \\ & - \eta h^2 \tilde{E}_p \sum_{\ell_1=0}^m \beta_{\ell_1} \xi_{n-\ell_1-1} = \sum_{\ell_1=0}^m \alpha_{\ell_1} \delta_{n+1-\ell_1} \end{aligned} \quad (4.83)$$

where

$$\xi_k = \sum_{j=0}^k \omega_{k,j} \varphi_j. \quad (4.84)$$

From (4.84),

$$\sum_{\ell_2=0}^m \alpha_{\ell_2} \xi_{n-\ell_2} = \sum_{\ell_2=0}^m \beta_{\ell_2} \varphi_{n-\ell_2} \quad (4.85)$$

so that replacing n by $n-\ell_2$ in (4.83), multiplying by α_{ℓ_2} and summing yields

$$\begin{aligned} & (I - \xi h \tilde{A} - \eta h^2 \tilde{A}^2) \sum_{\ell_2=0}^m \sum_{\ell_1=0}^m \alpha_{\ell_2} \alpha_{\ell_1} \varphi_{n+1-\ell_1-\ell_2} \\ & - ((\xi h I + \eta h^2 \tilde{A}) \tilde{E}_p + \eta h^2 \tilde{E}_p \tilde{A}) \sum_{\ell_2=0}^m \sum_{\ell_1=0}^m \alpha_{\ell_2} \beta_{\ell_1} \varphi_{n-\ell_1-\ell_2} \\ & - \eta h^2 \tilde{E}_p \sum_{\ell_2=0}^m \sum_{\ell_1=0}^m \beta_{\ell_2} \beta_{\ell_1} \varphi_{n-\ell_1-\ell_2-1} \\ & = \sum_{\ell_1=0}^m \sum_{\ell_2=0}^m \alpha_{\ell_1} \alpha_{\ell_2} \delta_{n+1-\ell_1-\ell_2} \end{aligned} \quad (4.86)$$

Equation (4.86) is a finite-term recurrence and yields the following result.

Theorem 4.7 (internal block-stability without auxiliary variables)

The recurrence (4.86) is stable if

$$\begin{aligned} & \det[(\underline{I} - \lambda_0 h \underline{A})(\underline{I} - \lambda_1 h \underline{A}) \mu^2 \{\rho^*(\mu)\}^2 \\ & - \{(\lambda_0 + \lambda_1) h \underline{E}_p - \lambda_0 \lambda_1 h^2 (\underline{E}_p \underline{A} + \underline{A} \underline{E}_p)\} \rho^*(\mu) \sigma^*(\mu) \mu \\ & - \eta h^2 \{\sigma^*(\mu)\}^2 \underline{E}_p] \end{aligned} \quad (4.87)$$

has zeros of modulus at most unity, those of modulus unity being semi-simple. Thus, the mixed quadrature -RK/mixed quadrature -RK method, with $\{\rho^*, \sigma^*\}$ - reducible rules, displays internal stability when applied to (4.3) if and only if

$$\det[\underline{R}(\lambda_0 h) \underline{R}(\lambda_1 h)] \quad (4.88)$$

is von Neumann, where

$$\underline{R}(\lambda h) = (\underline{I} - \lambda h \underline{A}) \mu \rho^*(\mu) - \lambda h \underline{E}_p \sigma^*(\mu). \quad (4.89)$$

4.4.2 Block-reducible rules

Employing the notation of §4.3.2 we may write (4.80) and (4.81) in the forms

$$\begin{aligned} & (\underline{I} \otimes (\underline{I} - \xi h \underline{A})) \hat{\varphi}_k - \xi h \sum_{j=0}^k (\hat{\underline{V}}_{kj} \otimes \underline{E}_p) \hat{\varphi}_j \\ & - \eta h (\underline{I} \otimes \underline{A}) \hat{\psi}_k - \eta h \sum_{j=0}^k (\hat{\underline{V}}_{kj} \otimes \underline{E}_p) \hat{\psi}_j \\ & = \hat{\delta}_k \end{aligned} \quad (4.90)$$

and

$$\hat{\psi}_k = h (\underline{I} \otimes \underline{A}) \hat{\varphi}_k + h \sum_{j=0}^k (\hat{\underline{V}}_{kj} \otimes \underline{E}_p) \hat{\varphi}_j. \quad (4.91)$$

Substitution of (4.91) in (4.90) gives a version of (4.82):

$$\begin{aligned} & (\underline{I} \otimes (\underline{I} - \xi h \underline{A} - \eta h^2 \underline{A}^2)) \hat{\varphi}_k - \xi h \sum_{j=0}^k (\hat{\underline{V}}_{kj} \otimes \underline{E}_p) \hat{\varphi}_j \\ & - \eta h^2 \sum_{j=0}^k (\hat{\underline{V}}_{kj} \otimes (\underline{E}_p \underline{A} + \underline{A} \underline{E}_p)) \hat{\varphi}_j - \eta h^2 \sum_{j=0}^k (\hat{\underline{V}}_{kj} \otimes \underline{E}_p) \hat{\varphi}_j = \hat{\delta}_k \end{aligned} \quad (4.92)$$

where

$$\hat{\xi}_j = \sum_{i=0}^j (\hat{V}_{ji} \otimes \hat{E}_p) \hat{\varphi}_i. \quad (4.93)$$

From the preceding equation, on setting $k = n - \ell_1$, multiplying by $\hat{A}_{\ell_1} \otimes \hat{I}$ and summing over ℓ_1 we find the result

$$\begin{aligned} & \sum_{\ell_1=0}^m (\hat{A}_{\ell_1} \otimes (\hat{I} - \xi \hat{h} \hat{A} - \eta \hat{h}^2 \hat{A}^2)) \hat{\varphi}_{n-\ell_1} \\ & - \sum_{\ell_1=0}^{m+1} (\hat{B}_{\ell_1} \otimes (\xi \hat{h} \hat{E}_p + \eta \hat{h}^2 (\hat{E}_p \hat{A} + \hat{A} \hat{E}_p))) \hat{\varphi}_{n-\ell_1} \\ & - \eta \hat{h}^2 \sum_{\ell_1=0}^{m+1} (\hat{B}_{\ell_1} \otimes \hat{E}_p) \hat{\xi}_{n-\ell_1} = \sum_{\ell_1=0}^m (\hat{A}_{\ell_1} \otimes \hat{I}) \hat{\delta}_{n-\ell_1}. \end{aligned} \quad (4.94)$$

We also have

$$\sum_{\ell_2=0}^{m+1} (\hat{A}_{\ell_2} \otimes \hat{I}) \hat{\xi}_{n-\ell_2} = \sum_{\ell_2=0}^{m+1} (\hat{B}_{\ell_2} \otimes \hat{E}_p) \hat{\varphi}_{n-\ell_2} \quad (4.95)$$

but in general this equation cannot be employed to eliminate $\hat{\xi}_{n-\ell_1}$ from (4.92). The two equations (4.92) and (4.93) then provide a basis for studying internal stability with auxiliary variables (the components of $\hat{\xi}_j$ being approximations to values of $z(x)$).

On the other hand, if the matrices \hat{A}_{ℓ_2} commute with the matrices \hat{B}_{ℓ_1} then we can obtain a relation not involving the vectors $\hat{\xi}_j$.

The required conditions obtain when $\hat{A}_0 = \hat{I}$, $\hat{A}_1 = -\hat{I}$, $\hat{A}_{\ell} = \hat{0}$,

$\ell = 2, 3, \dots, m$. Such conditions are associated with a

subclass of block reducible rules, namely those possessing

Property 5.1 of [4]. We illustrate with the following result, which subsumes the case of (ρ^*, σ^*) -reducibility (Theorem 4.7). Recall that

$$\hat{A}_{m+1} = \hat{0}.$$

Theorem 4.8 (block-stability for a subclass of block reducible rules)

Suppose that $\hat{A}_0 = \hat{I}$, $\hat{A}_1 = -\hat{I}$ and $\hat{A}_{\ell} = \hat{0}$, $\ell = 2, 3, \dots, m$, or more generally that the matrices \hat{A}_{ℓ_1} , \hat{B}_{ℓ_2} commute.

Then, using (4.95) in (4.94)

$$\begin{aligned}
 & \sum_{\ell_2=0}^{m+1} \sum_{\ell_1=0}^{m+1} [(\underline{A}_{\ell_2} \underline{A}_{\ell_1} \otimes (\underline{I} - \xi h \underline{A} - \eta h^2 \underline{A}^2) \\
 & - (\underline{A}_{\ell_2} \hat{\underline{B}}_{\ell_1}) \otimes (\underline{E}_p (\xi h \underline{I} + \eta h^2 \underline{A}) + \eta h^2 \underline{A} \underline{E}_p) \\
 & - \eta h^2 (\hat{\underline{B}}_{\ell_1} \hat{\underline{B}}_{\ell_2}) \otimes \underline{E}_p \hat{\varphi}_{n-\ell_1-\ell_2}] \quad (4.96) \\
 & = \hat{\delta}_n^*
 \end{aligned}$$

for an appropriate vector $\hat{\delta}_n^*$ and the associated stability polynomial is obtained on replacing $\hat{\varphi}_{n-\ell_1-\ell_2}$ by $\mu^{2m-\ell_1-\ell_2}$ in the left hand side and taking its determinant.

We conclude that the mixed quadrature -RK/mixed quadrature -RK method displays internal stability (without auxiliary variables) when applied to (4.3) if and only if

$$\det[\lambda_0 \underline{V}(\lambda_0 h) \underline{V}(\lambda_1 h) - \lambda_1 \underline{V}(\lambda_1 h) \underline{V}(\lambda_0 h)] \quad (4.97)$$

is von Neumann, where

$$\underline{V}(\lambda h) = \sum_{\ell=0}^{m+1} (\underline{A}_{\ell} \otimes \mu(\underline{I} - \lambda h \underline{A}) - \lambda h (\hat{\underline{B}}_{\ell} \otimes \underline{E}_p)) \mu^{m-\ell} \quad (4.98)$$

4.5 A-stability

We showed in [5] that mixed quadrature-Runge-Kutta methods for (4.1) need not be A-stable when both the associated quadrature method and extended Runge Kutta method are A-stable. The corresponding feature is present in the discussion of (4.3).

Definition 4.3

Let $\lambda_{0,1}$ be defined by (4.33). Then a method for (4.3) is said to be A-stable when it is strictly stable whenever $\text{Re}(\lambda_0 h) < 0$ and $\text{Re}(\lambda_1 h) < 0$. It is said to be A₀-stable when it is strictly stable whenever $\lambda_0 h < 0$ and $\lambda_1 h < 0$, $\lambda_0, \lambda_1 \in \mathbb{R}$.

Example 4.4 (a) With the trapezium rule and Runge-Kutta tableau

employed in Example 3.2 of [5] the RK/mixed quadrature RK method considered in §4.3.1 is A-stable. (b) With the quadrature rules defined by the backward Euler method, and the same Runge-Kutta tableau, the method considered in §4.3.1 is not A-stable and hence not A₀-Stable.

To investigate the method in Example 4.4 (b) we apply Theorem 4.5 with $\rho^*(\mu) = \mu - 1$, $\sigma^*(\mu) = \mu$, $\hat{\mu}(\lambda h) = (1 + \frac{1}{2}\lambda h)/(1 - \frac{1}{2}\lambda h)$ and $\nu_0 = \{(1 - \frac{1}{2}\lambda_0 h)(1 - \frac{1}{2}\lambda_1 h)\}^{-1} \cdot (1 + \frac{1}{2}\lambda_0 h + \frac{1}{2}\lambda_1 h - \frac{1}{4}\lambda_0 \lambda_1 h^2)$ and obtain the stability polynomial

$$(1 - \frac{1}{2}\lambda_0 h)(1 - \frac{1}{2}\lambda_1 h)\mu^2 + (\lambda_0 \lambda_1 h^2 - 2)\mu + [1 + \frac{1}{2}\lambda_0 h + \frac{1}{2}\lambda_1 h - \frac{1}{4}\lambda_0 \lambda_1 h^2].$$

Restricting attention to the case $\lambda_0, \lambda_1 < 0$, $h > 0$, this can be shown (by well-known techniques) to be a Schur polynomial only if

$$-4 < \eta h^2 < 0, \quad \xi h < 2$$

where $\eta = -\lambda_0 \lambda_1$, $\xi = \lambda_0 + \lambda_1$, and hence the method is not A-stable.

The corresponding extended method of §4.2.2 is A-stable.

Example 4.4 (b) established on analogue of Theorem 3.9 of [5], and prompts the search [7] for modified Runge-Kutta methods which possess the economy of mixed methods yet the desirable stability properties of extended methods.

4.6 Extension

The results presented in this chapter provide initial results extending those of [5] to the treatment of integro-differential equations. However, they do not exhaust the variations of Runge-Kutta methods available for the treatment of integro-differential equations. Additional investigations are pursued in chapter 5 where their results are presented and a general analysis is made of the stability polynomials derived so far.

REFERENCES

1. AMINI, S., BAKER, C.T.H. & WILKINSON, J.C. Basic stability analysis of Runge-Kutta methods for Volterra integral equations of the second kind. University of Manchester, Numer. Anal. Tech. Rept. No.46 (1980).
2. BAKER, C.T.H. Initial value problems for Volterra integro-differential equations. Chp. 21 in Modern numerical methods for ordinary differential equations (eds. HALL, G. & WATT, J.M.) Clarendon, Oxford (1976).
3. BAKER, C.T.H. The numerical treatment of integral equations, Clarendon Press, Oxford (1977, reprinted 1978).
4. BAKER, C.T.H., MAKROGLOU, A. & SHORT, E. Regions of stability in the numerical treatment of Volterra integro-differential equations. Siam. J. Numer. Anal. Vol 16, No.6 (1979).
5. BAKER, C.T.H. & WILKINSON, J.C. Stability analysis of Runge-Kutta methods applied to a basic Volterra integral equation. J. Austral. Math. Soc. (Series B) 22 (1981) pp.515-538.
6. BAKER, C.T.H. & WILKINSON, J.C. Stability of Runge-Kutta methods applied to a basic integro-differential equation, Manuscript, October 1981 (A revised version of Numer. Anal. Tech. Rept. No.50, University of Manchester, June 1980).
7. BAKER, C.T.H. & WILKINSON, J.C. On the construction of stability polynomials for modified R-K methods for Volterra integro-differential equations. Contribution to Treatment of Integral Equations by Numerical Methods (eds BAKER, C.T.H. & MILLER, G.F.), Academic Press (1982). Also Num. Anal. Tech. Rept. No.81, University of Manchester, September 1982.
8. BRUNNER, H. & LAMBERT, J. Stability of numerical methods for Volterra integro-differential equations. Computing, 12, (1974) pp.75-84.
9. FELDSTEIN, A. & SOPKA, J.R. Numerical methods for non-linear Volterra integro-differential equations. SIAM J. Numer. Anal, 11, (1974) pp826-846.
10. HALL, G. & WATT, J.M. (eds) Modern numerical methods for ordinary differential equations. Clarendon Press, Oxford (1976).

11. von der HOUWEN, P.J., TE RIELE, H.J.J. & WOLKENFELT, P.H.M. On the stability of multistep formulas for systems of Volterra integro-differential equations. Rep NW 63/78, Mathematisch Centrum, Amsterdam (1978).
12. van der HOUWEN, P.J., WOLKENFELT, P.H.M. & BAKER, C.T.H. Coverage and stability analysis for modified Runge-Kutta methods in the numerical treatment of second-kind Volterra integral equations. IMA J.Numer.Anal. 1 (1981) pp303-328.
13. LINZ, P. Linear Multistep methods for Volterra integro-differential equations. J. Ass. Comput. Mach 16 (1969) pp295-301.
14. MAKROGLOU, A. Numerical solution of Volterra integro-differential equations. Ph.D. Thesis, University of Manchester, 1977.
15. MATTYS, J. A-stable linear multistep methods for Volterra integro-differential equations. Numer. Math. 27 (1976) pp 85-94.
16. MCKEE, S. Cyclic multistep methods for solving Volterra integro-differential equations. SIAM J. Numer. Anal. 16 (1979) pp106-114.
17. MOCARSKÝ, W.L. Convergence of step-by-step methods for non-linear integro-differential equations. J. Inst. Math. Appl, 8, (1971) pp235-239.
18. TAVERNINI, L. One step methods for the numerical solution of Volterra functional differential equations. SIAM J. Numer. Anal, 8, (1971) pp786-795.
19. WOLKENFELT, P.H.M. Linear multistep methods and the construction of quadrature formulae for Volterra integral and integro-differential equations. Rep. NW 76/79 Mathematisch Centrum. Amsterdam (1979).

CHAPTER 5

5. On the construction of stability polynomials for modified R-K
methods for Volterra integro-differential equations.

- 5.1 Introduction
- 5.2. Stability of a recurrence
- 5.3 A general class of methods
- 5.4 Basic stability theory
- 5.5 Summary

5.1 Introduction

In this chapter we continue our interest in the stability analysis of a wide class of numerical methods for the Volterra integro-differential equation

$$f'(x) = G(x, f(x), \int_0^x H(x, y, f(y)dy) + d(x) \quad (x \geq 0). \quad (5.1)$$

The results are applicable due to the choice as a stability test equation of the "basic" integro-differential equation

$$f'(x) = \xi f(x) + \eta \int_0^x f(t)dt + d(x) \quad (5.2)$$

and because we assume reducibility structure [1, 6] in the numerical methods. The methods covered include classical methods treated in [3] and described in chapter 4 of this manuscript and new γ -modified Runge-Kutta methods which are motivated by the corresponding methods, first introduced, for Volterra integral equations of the second kind, by Van der Houwen [5]. The γ -modified R-K methods are introduced in §2.6 of this work.

Our purpose is to gain insight into the construction of stability polynomials for classical and modified Runge-Kutta methods which we describe below. This task is, in our view, a prerequisite to further study. Although (5.2) reduces, on differentiation, to a second-order differential equation the approach to our methods is from the viewpoint of a system of integral equations and the modified methods are of particular interest. Incorporating the modified methods into a general analysis requires some rather special insight which we elucidate below (cf. [4]).

5.2 Stability of a recurrence

We commence by investigating, briefly, recurrence relations of the form

$$\sum_{\ell=0}^{m+1} \tilde{P}_{\ell} \hat{\varphi}_{n+1-\ell} + \sum_{\ell=0}^{m+1} \tilde{Q}_{\ell} \hat{\psi}_{n+1-\ell} = \tilde{\Delta}'_{n+1} \quad (5.3)$$

$$\sum_{\ell=0}^{m+1} \tilde{R}_{\ell} \hat{\varphi}_{n+1-\ell} + \sum_{\ell=0}^{m+1} \tilde{S}_{\ell} \hat{\psi}_{n+1-\ell} = \tilde{\Delta}''_{n+1}. \quad (5.4)$$

Hence $(\tilde{P}_{\ell}, \tilde{Q}_{\ell}, \tilde{R}_{\ell}, \tilde{S}_{\ell})$ are square matrices of fixed order and $(\hat{\varphi}_n, \hat{\psi}_n)_{n \geq 0}$ is a sequence of vector pairs. Considering (5.3) and (5.4) we have the following result (cf. [3]).

Theorem 5.1

A stability polynomial for (5.3) and (5.4) is

$$\Sigma(\mu) \equiv \det \begin{bmatrix} \tilde{P} & \tilde{Q} \\ \tilde{R} & \tilde{S} \end{bmatrix} \quad (5.5)$$

where $\tilde{P} \equiv \tilde{P}(\mu) = \sum_{\ell=0}^{m+1} \tilde{P}_{\ell} \mu^{m+1-\ell}$, and likewise for $\tilde{Q} \equiv \tilde{Q}(\mu)$,

$\tilde{R} \equiv \tilde{R}(\mu)$ and $\tilde{S} \equiv \tilde{S}(\mu)$.

Proof

It is convenient to express the recurrence relation (5.3) and (5.4) as follows

$$\begin{bmatrix} \tilde{P}_0 & \circ & \tilde{Q}_0 & \circ \\ \circ & \tilde{I} & \circ & \circ \\ \tilde{R}_0 & \circ & \tilde{S}_0 & \circ \\ \circ & \circ & \circ & \tilde{I} \end{bmatrix} \begin{bmatrix} \varphi_{n+1} \\ \vdots \\ \varphi_{n+1-m} \\ \psi_{n+1} \\ \vdots \\ \psi_{n+1-m} \end{bmatrix} =$$

$$\begin{bmatrix} \tilde{P}_1 & \dots & \tilde{P}_{m+1} & \tilde{Q}_1 & \dots & \tilde{Q}_{m+1} \\ \vdots & & \vdots & & & \vdots \\ \tilde{I} & & \tilde{Q} & \tilde{Q} & \tilde{Q} & \vdots \\ & & \vdots & & & \vdots \\ \vdots & & \vdots & & & \vdots \\ \tilde{R}_1 & \dots & \tilde{R}_{m+1} & \tilde{S}_1 & \dots & \tilde{S}_{m+1} \\ \vdots & & \vdots & & & \vdots \\ \tilde{Q} & \tilde{Q} & \tilde{I} & \tilde{Q} & \tilde{Q} & \vdots \\ & & \vdots & & & \vdots \\ \vdots & & \vdots & & & \vdots \\ \tilde{\psi}_n & \dots & \tilde{\psi}_{n-m} & \tilde{\psi}_n & \dots & \tilde{\psi}_{n-m} \end{bmatrix} + \begin{bmatrix} \tilde{\Delta}'_{n+1} \\ \vdots \\ \tilde{\Delta}''_{n+1} \\ \vdots \end{bmatrix} \quad (5.6)$$

This may be simplified if the following procedures are observed.

1. Add $\mu \times$ column s to column $(s+1)$ for successively increasing values of s from $s=1$ to $s=m$.
2. Add $\mu \times$ row r to row $(r-1)$ for successively decreasing values of r from $r = 2m+2$ to $r = m+3$.
3. To column $2m+2$ add $\sum_{\ell=1}^m \mu^\ell \times$ column $(2m+2-\ell)$.

A simplified form of the stability polynomial may now be expressed from which the required result is readily derived

$$\det \begin{bmatrix} \begin{array}{c|c} -\tilde{P}_1 - \mu \tilde{P}_0 & \dots & \sum_{\ell=0}^{m+1} \tilde{P}_\ell \mu^{m+1-\ell} \\ \hline \tilde{I} & & \circ & & \circ & & \circ & & \circ \end{array} & \begin{array}{c|c} -\tilde{Q}_1 - \mu \tilde{Q}_0 & \dots & \sum_{\ell=0}^{m+1} \tilde{Q}_\ell \mu^{m+1-\ell} \\ \hline \circ & & \circ & & \circ & & \circ & & \circ \end{array} \\ \hline \begin{array}{c|c} -\tilde{R}_1 - \mu \tilde{R}_0 & \dots & \sum_{\ell=0}^{m+1} \tilde{R}_\ell \mu^{m+1-\ell} \\ \hline \circ & & \circ & & \circ & & \circ & & \circ \end{array} & \begin{array}{c|c} -\tilde{S}_1 - \mu \tilde{S}_0 & \dots & \sum_{\ell=0}^{m+1} \tilde{S}_\ell \mu^{m+1-\ell} \\ \hline \circ & & \circ & & \circ & & \circ & & \circ \end{array} \end{bmatrix}$$

(5.8)

In general we wish to expand (5.5). We have a number of possibilities of which the following are illustrative (but not exhaustive).

Case 1 $\tilde{S}\tilde{Q} = \tilde{Q}\tilde{S}$ and \tilde{S}^{-1} exists then $\sum(\mu) = \det [\tilde{S}\tilde{P} - \tilde{Q}\tilde{R}]$.

Note that

$$\begin{bmatrix} \tilde{P} & \tilde{Q} \\ \tilde{R} & \tilde{S} \end{bmatrix} \begin{bmatrix} \tilde{I} & 0 \\ -\tilde{S}^{-1}\tilde{R} & \tilde{I} \end{bmatrix} = \begin{bmatrix} \tilde{P} - \tilde{Q}\tilde{S}^{-1}\tilde{R} & \tilde{Q} \\ \tilde{R} & \tilde{S} \end{bmatrix} = \begin{bmatrix} \tilde{P} - \tilde{S}^{-1}\tilde{Q}\tilde{R} & \tilde{Q} \\ \tilde{R} & \tilde{S} \end{bmatrix}$$

Case 2 $\underline{P}\underline{R} = \underline{R}\underline{P}$ and \underline{P}^{-1} exists then $\sum(\mu) = \det [\underline{P}\underline{S} - \underline{R}\underline{Q}]$.

Note that
$$\begin{bmatrix} \underline{P} & \underline{Q} \\ \underline{R} & \underline{S} \end{bmatrix} \begin{bmatrix} \underline{I} & -\underline{P}^{-1}\underline{Q} \\ \underline{0} & \underline{I} \end{bmatrix} = \begin{bmatrix} \underline{P} & \underline{0} \\ \underline{R} & \underline{S} - \underline{R}\underline{P}^{-1}\underline{Q} \end{bmatrix} = \begin{bmatrix} \underline{P} & \underline{0} \\ \underline{R} & \underline{S} - \underline{P}^{-1}\underline{R}\underline{Q} \end{bmatrix}$$

Case 3 If \underline{S}^{-1} exists then $\sum(\mu) = \det [\underline{P} - \underline{Q}\underline{S}^{-1}\underline{R}] \det \underline{S}$.

For remarks on Case 3 see [2]. We observe that Cases 1 and 2 (and, indeed, Case 3) correspond to elimination of either $\{\hat{\psi}_n\}$ or $\{\hat{\varphi}_n\}$ between (5.3) and (5.4). Thus, for example, if we denote by E the advancement operator ($E\hat{\varphi}_n = \hat{\varphi}_{n+1}$, $E\hat{\psi}_n = \hat{\psi}_{n+1}$) then (5.3) and (5.4) become

$$\underline{P}(E)\hat{\varphi}_{n-m} + \underline{Q}(E)\hat{\psi}_{n-m} = \underline{\Delta}'_{n+1} \quad (5.9)$$

$$\underline{R}(E)\hat{\varphi}_{n-m} + \underline{S}(E)\hat{\psi}_{n-m} = \underline{\Delta}''_{n+1} \quad (5.10)$$

where $\underline{P}(E)$ denotes substitution of E for μ in $\underline{P}(\mu)$ etc.

Apply $\underline{S}(E)$ to (5.9) and $\underline{Q}(E)$ to (5.10) with some loss of information if either \underline{S} or \underline{Q} is singular. Subtracting now yields, if we make the assumption that $\underline{Q}(E)\underline{S}(E) = \underline{S}(E)\underline{Q}(E)$,

$$[\underline{S}(E)\underline{P}(E) - \underline{Q}(E)\underline{R}(E)]\hat{\varphi}_{n-m} = \underline{S}(E)\underline{\Delta}'_{n+1} - \underline{Q}(E)\underline{\Delta}''_{n+1} \quad (5.11)$$

for which a stability polynomial is Case 1, $\det[\underline{S}(\mu)\underline{P}(\mu) - \underline{Q}(\mu)\underline{R}(\mu)]$.

5.3 A General Class of Methods

We recall from §4.1.2 that if we write (5.1) in integrated form, we obtain the pair of integral equations

$$f(x) = \int_0^x G(s, f(s), z(s)) ds + d(x), \quad (5.12a)$$

$$z(x) = \int_0^x H(x, t, f(t)) dt, \quad (5.12b)$$

and we seek approximations $\tilde{f}_j = f(\tau_j)$ and $\tilde{z}_j = z(\tau_j)$ at the points

$$\tau_j = ih + \theta_r h \quad (i=0, 1, 2, \dots; r=0, 1, \dots, p; j=i(p+1)+r+1).$$

Here, the values θ_r are parameters defined by a Runge-Kutta method.

We require quadrature rules with abscissae $\{\tau_j\}$ of the form

$$\hat{Q} : \int_0^{\tau_j} \varphi(t) dt \approx h \sum_{k=0}^{(i+1)(p+1)} \Omega_{jk} \varphi(\tau_k), \quad (5.13)$$

with weights $\hat{Q}' = \{\Omega'_{jk}\}$, $\hat{Q}'' = \{\Omega''_{jk}\}$. We define $\Omega'_{jk} = \Omega''_{jk} = 0$, for $k > (i+1)(p+1)$. A choice of vectors

$$\underline{\gamma}' = (\gamma'_0, \dots, \gamma'_p)^T \quad (5.14')$$

$$\underline{\gamma}'' = (\gamma''_0, \dots, \gamma''_p)^T \quad (5.14'')$$

with components in $[0,1]$ permits the replacement of (5.12) by the equations given below (cf. §2.6 equation (2.83) and (2.84))

$$\begin{aligned} \tilde{f}_j &= h \sum_{k \geq 0} \Omega'_{jk} G(\tau_k, \tilde{f}_k, \tilde{z}_k) + d(\tau_j) \\ &+ \gamma'_r (\tilde{f}_{i(p+1)} - (h \sum_{k=0}^{i(p+1)} \Omega'_{i(p+1)+1,k} G(\tau_k, \tilde{f}_k, \tilde{z}_k) + d(ih))) \end{aligned} \quad (5.15)$$

and

$$\begin{aligned} \tilde{z}_j &= h \sum_{k \geq 0} \Omega''_{jk} H(\tau_j, \tau_k, \tilde{f}_k) \\ &+ \gamma''_r (\tilde{z}_{i(p+1)} - (h \sum_{k=0}^{i(p+1)} \Omega''_{i(p+1)+1,k} H(ih, \tau_k, \tilde{f}_k))) \end{aligned} \quad (5.16)$$

The corresponding method will be denoted $M(\Omega', \underline{\gamma}'; \Omega'', \underline{\gamma}'')$. These equations with $\gamma'_r = \gamma''_r = 0$ ($r = 0, 1, \dots, p$) define classical methods which are modified by the introduction of the parameters (5.14'), (5.14''). (The more natural class of methods requires only the choice of $\underline{\gamma}''$, with $\underline{\gamma}' = \underline{0}$).

It remains to construct the rules (\hat{Q}') and (\hat{Q}'') , of the form (5.13). We shall employ R-K parameters $[\underline{\theta} | \underline{A}]$ defined by the tableau

in (2.33) and the quadrature rules Q defined in (2.3) with weights $\{\omega_{ik}\}$ ($\omega_{ik} = 0, k > i$). Subsequently we shall employ rules Q', Q'' with weights $\omega'_{ik}, \omega''_{ik}$ and R-K tableaux $[\underline{\varrho}', \underline{A}']$ and $[\underline{\varrho}'', \underline{A}'']$ with $\underline{\varrho}' = \underline{\varrho}'' = \underline{\varrho}$. Rules Q and tableaux $[\underline{\varrho}, \underline{A}]$ permit the definition of \hat{Q} .

The extended Runge-Kutta method uses 'weights' denoted by $\Omega_{jk}(\underline{A})$ which are defined in (2.53). We take $\underline{\gamma} = \underline{0}$ for definiteness and write $\hat{Q} = \Omega(\underline{A})$ for use in (5.13).

The mixed Runge-Kutta methods uses 'weights' denoted by $\Omega_{jk}[Q; \underline{A}]$ which are defined in (2.54) and we write $\hat{Q} = \Omega[Q; \underline{A}]$. Thus, the weights Ω'_{jk} and Ω''_{jk} may be defined as

$$\Omega'_{jk} = \Omega_{jk}(\underline{A}') \quad (5.17')$$

or

$$\Omega'_{jk} = \Omega_{jk}[Q'; \underline{A}'] \quad (5.18')$$

and

$$\Omega''_{jk} = \Omega_{jk}(\underline{A}'') \quad (5.17'')$$

or

$$\Omega''_{jk} = \Omega_{jk}[Q''; \underline{A}''] \quad (5.18'')$$

When $\Omega_{jk}(\underline{A})$ are taken as the weights defining Ω' or Ω'' , the corresponding $\underline{\gamma}(\underline{\gamma}' \text{ or } \underline{\gamma}'')$ may be chosen arbitrarily. Computationally one would assume $\underline{\gamma} = \underline{0}$ but it is convenient in §5.4, to suppose that $\underline{\gamma} = (1, 1, \dots, 1)^T$. When $\Omega'_{jk} = \Omega_{jk}(\underline{A})$, the method may be regarded as an adaptation of an R-K method for an ordinary differential equation.

Furthermore, we require the weights Ω_{jk} to have structure. We recall, from Definition 2.5, that the array of weights $\{\bar{\Omega}_{jk}\}$ is block reducible when the weights $\bar{\Omega}_{jk}$ may be partitioned into matrices $\bar{\Omega}_{nk}$ of order q with elements $\varepsilon_{\alpha}^T \bar{\Omega}_{nk} \varepsilon_{\beta} = \bar{\Omega}_{nq+\alpha, kq+\beta}$ ($\alpha, \beta = 0, 1, \dots, q-1$) ($\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{q-1}$ are the columns of the identity matrix of order q). The matrices $\bar{\Omega}_{nk}$ satisfy, for fixed matrices $\{\bar{A}_{\varrho}, \bar{B}_{\varrho}\}_{\varrho=0}^m$, where $\sum_{\varrho=0}^m \bar{A}_{\varrho} \varepsilon = \underline{0}$, the relation

$$\sum_{\ell=0}^m \bar{A}_{\ell}, \bar{V}_{n-\ell, k} = \bar{B}_{n-k}, \quad (5.19)$$

with the convention $\bar{B}_{\ell} = 0$ for $\ell \notin \{0, 1, \dots, m\}$. If $\bar{A}_0 = I$, $\bar{A}_1 = -I$ in (5.19), $\bar{A}_m = 0$ otherwise, \bar{Q} is termed simply-block-reducible. If $q=1$ in (5.19), \bar{Q} is called reducible.

'Reducibility', in one of the above senses will be assumed in the sequel. 'Reducibility' of Q yields 'reducibility' of the rules with weights $\Omega_{jk} [Q; \bar{A}]$, under an associated partitioning.

5.4 Basic Stability Theory

Our purpose is to discuss the stability of a set of numerical methods, included in the description in §5.3, when they are applied to the basic test equation (5.2). We shall employ the notation, defined in (4.23) and (4.24), viz.

$$\varphi_{n+1} = [\tilde{f}_{n(p+1)+1}, \dots, \tilde{f}_{(n+1)(p+1)}]^T,$$

$$\psi_{n+1} = [\tilde{z}_{n(p+1)+1}, \dots, \tilde{z}_{(n+1)(p+1)}]^T$$

where $\{\tilde{f}_n\}$, $\{\tilde{z}_n\}$ are the values obtained for equation (5.2). We also write $\tilde{\Gamma} = \text{DIAG} (\gamma_0, \gamma_1, \dots, \gamma_p)$, $\tilde{\Gamma}'$, $\tilde{\Gamma}''$ being defined in terms of γ' , γ'' . It is also convenient to write

$$\tilde{\Gamma}'_p = \tilde{\Gamma}' E_p, \quad (5.20')$$

$$\tilde{\Gamma}''_p = \tilde{\Gamma}'' E_p. \quad (5.20'')$$

The notation φ_n , ψ_n will be used in a generic sense. Since we have to consider various methods applied to (5.2), we write

$\varphi_{n+1}^{(e)}$ for vectors generated using $\Omega_{jk}(\bar{A}')$, $\gamma' = 0$ in (5.15);

$\psi_{n+1}^{(e)}$ " " " " $\Omega_{jk}(\bar{A}'')$, $\gamma'' = 0$ in (5.16),

(the "extended" Runge-Kutta formulae)

and

φ'_{n+1} for vectors generated using $\Omega_{jk}[Q'; \tilde{A}']$, γ' in (5.15);

ψ''_{n+1} for vectors generated using $\Omega_{jk}[Q''; \tilde{A}'']$, γ'' in (5.16),

(the " γ' - and γ'' - modified mixed formulae").

Thus, we find that (5.17') and (5.2) yield, from (5.15)

$$\varphi_{n+1}^{(e)} = E_p \varphi_n^{(e)} + h \tilde{A}' (\xi \varphi_{n+1}^{(e)} + \eta \psi_{n+1}) \quad (5.21)$$

and (5.18') and (5.2) yield from (5.15)

$$\varphi'_{n+1} = \varphi_{n+1}^\dagger + \Gamma_p' (\varphi_n' - h \sum_{j=0}^n \omega_{nj}' (\xi \varphi_j' + \eta \psi_j)) \quad (5.22a)$$

wherein

$$\varphi_{n+1}^\dagger = h \sum_{j=0}^n \omega_{nj}' E_p (\xi \varphi_j' + \eta \psi_j) + h \tilde{A}' (\xi \varphi_{n+1}' + \eta \psi_{n+1}). \quad (5.22b)$$

Similarly $\psi_{n+1}^{(e)} = h \sum_{j=0}^{n-1} E_p \tilde{A}'' \varphi_{j+1} + h \tilde{A}'' \varphi_{n+1}$ whence

$$\psi_{n+1}^{(e)} = E_p \psi_n^{(e)} + h \tilde{A}'' \varphi_{n+1} \quad (5.23)$$

$$\text{whilst } \psi''_{n+1} = \psi_{n+1}^\dagger + \Gamma_p'' (\psi_n'' - h \sum_{j=0}^n \omega_{nj}'' E_p \varphi_j) \quad (5.24a)$$

$$\text{wherein } \psi_{n+1}^\dagger = h \sum_{j=0}^n \omega_{nj}'' E_p \varphi_j + h \tilde{A}'' \varphi_{n+1}. \quad (5.24b)$$

Our task is to analyse the stability of (5.21) or (5.22a,b) where (ψ_n) is replaced by $(\psi_n^{(e)})$ or (ψ_n'') . In order to analyse the possible combination of (5.21) or (5.22) with (5.23) or (5.24) we require some common structure which we shall develop. In particular we ask, when dealing with (5.22) or (5.24), that the rules $\{\omega_{nj}'\}$ or $\{\omega_{nj}''\}$ should be block-reducible. Then we require the matrices (V_{nj}') , (V_{nj}'') of order q' , q'' respectively with the elements

$$\varepsilon_\alpha'^T V_{nj}' \varepsilon_\beta' = \omega_{nq'+\alpha, jq'+\beta} \quad (\alpha, \beta = 0, 1, \dots, q'-1) \quad (5.25')$$

$$\varepsilon_\alpha''^T V_{nj}'' \varepsilon_\beta'' = \omega_{nq''+\alpha, jq''+\beta} \quad (\alpha, \beta = 0, 1, \dots, q''-1) \quad (5.25'')$$

in the (α, β) position, respectively. Associated with q', q'' (where appropriate) we define a parameter q which assumes, in our analysis of the indicated pair of equations, the value shown

	Eq. (5.21) ($\varphi_n^{(e)}$)	Eq. (5.22) (φ_n')
Eq. (5.23) ($\psi_n^{(e)}$)	$q=1$	$q=q'$
Eq. (5.24) (ψ_n'')	$q=q''$	$q=q'=q''$

Table 5.1

Observe that in our analysis of (5.22) and (5.24) we require $q' = q''$ but if this is not the case we are able by restructuring to take $q = q'q''$. Recalling the notation \underline{J} and $\underline{J}^\#$ defined in §4.3.2, we derive from the matrices \underline{V}_{nj}' and \underline{V}_{nj}'' , as appropriate, the matrices $\hat{\underline{V}}_{nj}', \hat{\underline{V}}_{nj}''$ from the rule

$$\hat{\underline{V}}_{nj} = \underline{V}_{nj} \underline{J} + \underline{V}_{n,j+1} \underline{J}^\#. \quad (5.26)$$

Having established our interpretation of q , we introduce the vectors $\hat{\varphi}_n^{(e)}, \hat{\varphi}_n', \hat{\psi}_n^{(e)}, \hat{\psi}_n''$ obtained on replacing φ_n and ψ_n in an obvious manner in the notation, first introduced in (4.68),

$$\hat{\varphi}_{n+1} = (\varphi_{nq+1}^T, \varphi_{nq+2}^T, \dots, \varphi_{(n+1)q}^T)^T \quad (5.27)$$

$$\hat{\psi}_{n+1} = (\psi_{nq+1}^T, \psi_{nq+2}^T, \dots, \psi_{(n+1)q}^T)^T \quad (5.28)$$

($n=0,1,2,\dots$) with $\hat{\varphi}_0 = f(0)\{0,0,\dots,0,1\}^T$, $\hat{\psi} = \underline{0} \in \mathbb{C}^{q(p+1)}$

Finally, recalling that the Kronecker product $\underline{G} \otimes \underline{H}$ of the square matrices \underline{G} and \underline{H} is the partitioned matrix with elements $G_{\alpha\beta}H_{\gamma\delta}$, we can state the following results.

Lemma 5.1

If (a) $\hat{\varphi}_n = \hat{\varphi}_n^{(e)}$ in (5.27) and $\Gamma'_p = E_p$, \hat{V}_{nj} arbitrary, or
 (b) $\hat{\varphi}_n = \hat{\varphi}_n'$ in (5.27) and $\Gamma'_p = \Gamma'_p E_p$, \hat{V}_{nj} satisfying (5.25'),
 then (a) (5.21) or (b) (5.22) yields

$$\begin{aligned} \{I_q \otimes [I - \xi h A']\} \hat{\varphi}_{n+1} &= \xi h \sum_{j=0}^{n+1} \hat{V}_{n+1,j}' \otimes (E_p - \Gamma'_p) \hat{\varphi}_j + (I_q \otimes \Gamma'_p) \hat{\varphi}_n + \\ &+ \eta h \sum_{j=0}^{n+1} \hat{V}_{n+1,j}' \otimes (E_p - \Gamma'_p) \hat{\psi}_j + \eta h (I_q \otimes A') \hat{\psi}_{n+1} + \hat{\delta}_{n+1}. \end{aligned} \quad (5.29)$$

Proof

(a) The analogue of (5.21) for $\hat{\varphi}_n^{(e)}$ is

$$I_q \otimes \hat{\varphi}_{n+1}^{(e)} = (I_q \otimes E_p) \hat{\varphi}_n^{(e)} + h(I_q \otimes A') \{ \xi \hat{\varphi}_{n+1}^{(e)} + \eta \hat{\psi}_{n+1} \}$$

On re-arranging we obtain

$$\{I_q \otimes (I_q - \xi h A')\} \hat{\varphi}_{n+1}^{(e)} = (I_q \otimes E_p) \hat{\varphi}_n^{(e)} + \eta h (I_q \otimes A') \hat{\psi}_{n+1}.$$

This result has already been stated in (4.69) and may also be derived from (5.29) when $\Gamma'_p = E_p$.

(b) Rearranging (5.22) we obtain

$$\begin{aligned} (I - \xi h A') \hat{\varphi}_{n+1}' &= \xi h \sum_{j=0}^n \omega_{nj}' (E_p - \Gamma'_p) \hat{\varphi}_j' + \Gamma'_p \hat{\varphi}_n' + \eta h A' \hat{\psi}_{n+1} + \\ &+ \eta h \sum_{j=0}^n \omega_{nj}' (E_p - \Gamma'_p) \hat{\psi}_j. \end{aligned}$$

The analogue of (5.22) for $\hat{\varphi}_n'$ is

$$\begin{aligned} \{I_q \otimes (I - \xi h A')\} \hat{\varphi}_{n+1}' &= \xi h \sum_{j=0}^{n+1} \hat{V}_{n+1,j}' \otimes (E_p - \Gamma'_p) \hat{\varphi}_j' + (I_q \otimes \Gamma'_p) \hat{\varphi}_n' + \\ &+ \eta h (I_q \otimes A') \hat{\psi}_{n+1} + \eta h \sum_{j=0}^{n+1} \hat{V}_{n+1,j}' \otimes (E_p - \Gamma'_p) \hat{\psi}_j. \end{aligned}$$

This is the required result (5.29) which has already been derived in (4.90).

Lemma 5.2

- If (a) $\hat{\psi}_n = \hat{\psi}_n^{(e)}$ in (5.28) and $\Gamma_p'' = E_p$, \hat{V}_{nj}'' arbitrary or
 (b) $\hat{\psi}_n = \hat{\psi}_n''$ in (5.28) and $\Gamma_p'' = \Gamma_p'' E_p$, \hat{V}_{nj}'' satisfying (5.25'') then
 (a) (5.23) or (b) (5.24) yields

$$\hat{\psi}_{n+1} = h \sum_{j=0}^{n+1} \hat{V}_{n+1,j}'' \otimes (E_p - \Gamma_p'') \hat{\varphi}_j + h(I_q \otimes \hat{A}'') \hat{\psi}_{n+1} + (I_q \otimes \Gamma_p'') \hat{\psi}_n \quad (5.30)$$

Proof By re-arrangement of the analogues of (5.23) and (5.24).

We pause to recall, from §4.3.2, that when the quadrature rules $Q = \{\omega_{ij}\}$ are block-reducible with $\sum_{\ell=0}^m A_{\ell} V_{n-\ell,j} = B_{n-j}$ where $A_{\ell} = B_{\ell} = 0$ for $\ell \neq \{0, 1, \dots, m\}$ then $\sum_{\ell=0}^{m+1} A_{\ell} \hat{V}_{n-\ell,j} = \hat{B}_{n-j}$ where $\hat{B}_{\ell} = 0$ for $\ell \notin \{0, 1, \dots, m, m+1\}$ and

$$\hat{B}_{\ell} = B_{\ell-1} J^{\#} + B_{\ell} J. \quad (5.31)$$

The matrices \hat{B}_{ℓ}' , \hat{B}_{ℓ}'' are derived according to the rule (5.31), but when treating $\hat{\varphi}_n^{(e)}$ (respectively $\hat{\psi}_n^{(e)}$) the matrices \hat{V}_{nj}' (respectively \hat{V}_{nj}'') are arbitrary and we then set $A_0' = I$, $A_{\ell}' = 0$ otherwise (respectively $A_0'' = I$, $A_{\ell}'' = 0$ otherwise) \hat{B}_{ℓ}' and \hat{B}_{ℓ}'' being arbitrary.

Lemma 5.3

- (a) Let the rules $\{\omega_{nj}\}$ be block-reducible. Then (5.29) yields (5.3) where

$$P_{\ell} = A_{\ell}' \otimes (I - \xi h A') - \xi h B_{\ell}' \otimes (E_p - \Gamma_p') - A_{\ell-1}' \otimes \Gamma_p' \quad (5.32)$$

and $Q_{\ell} = -\eta h [\hat{B}_{\ell}' \otimes (E_p - \Gamma_p') + A_{\ell}' \otimes A']$. (5.33)

- (b) Let the rules $\{\omega_{nj}\}$ be block-reducible. Then (5.30) yields (5.4) where

$$R_{\ell} = -h [\hat{B}_{\ell}'' \otimes (E_p - \Gamma_p'') + A_{\ell}'' \otimes A''] \quad (5.34)$$

$$\text{and } \underline{S}_\ell = (\underline{A}_\ell'' \otimes \underline{I}) - (\underline{A}_{\ell-1}'' \otimes \underline{\Gamma}_p''). \quad (5.35)$$

Proof By applying $\underline{A}_\ell' \otimes \underline{I}_q$ to (5.29) and $\underline{A}_\ell'' \otimes \underline{I}_q$ to (5.30).

Observe that $\underline{P}_\ell \equiv \underline{P}_\ell(\xi h)$, $\underline{Q}_\ell \equiv \underline{Q}_\ell(\eta h)$, $\underline{R}_\ell \equiv \underline{R}_\ell(h)$ and \underline{S}_ℓ is independent of ξ , η and h . As earlier we write

$$\underline{P}(\mu) = \sum_{\ell=0}^{m+1} \underline{P}_\ell \mu^{m+1-\ell} \text{ etc in what follows.}$$

Our reference to equations (5.3) and (5.4) establishes the connection with Theorem 5.1. However, the result of Case 1 requires $\underline{S}\underline{Q}=\underline{Q}\underline{S}$, but in general the presence of $\underline{\Gamma}_p''$ causes difficulties with this condition even under the reasonable assumption that $\underline{A}_0'' = \underline{I}$, $\underline{A}_1'' = -\underline{I}$, $\underline{A}_\ell'' = \underline{Q}$ otherwise (simple block-reducibility). For the combination (5.21) with (5.23) we can appeal to Case 3 since \underline{S}^{-1} is readily obtained, but to effect a general treatment we proceed as follows.

Lemma 5.4

Under the assumption of Lemma 5.3 and with the notation established above,

$$\underline{P}(E) \hat{\underline{\varphi}}_{n-m} + \underline{Q}(E) \hat{\underline{\psi}}_{n-m} = \underline{\Delta}_{n+1}' \quad (5.36)$$

$$\text{and } \underline{R}^*(E) \hat{\underline{\varphi}}_{n-m} + \underline{S}^*(E) \hat{\underline{\psi}}_{n-m} = \underline{\Delta}_{n+1}''' \quad (5.37)$$

$$\text{where } \underline{R}^*(\mu) = [\underline{I}_q \otimes \{(\mu - \gamma_p'') \underline{I} + \underline{\Gamma}_p''\}] \underline{R}(\mu) \quad (5.38)$$

$$\begin{aligned} \underline{S}^*(\mu) &= [\underline{I}_q \otimes \{(\mu - \gamma_p'') \underline{I} + \underline{\Gamma}_p''\}] \underline{S}(\mu) \\ &= (\mu - \gamma_p'') \sum_{\ell=0}^{m+1} (\underline{A}_\ell'' \otimes \underline{I}) \mu^{m+1-\ell} \end{aligned} \quad (5.39)$$

and if $q=1$ or the rules $\underline{Q}'' = \{\omega_{nj}''\}$ are simply block-reducible, \underline{S}^* commutes with \underline{Q} .

Proof

Equation (5.4) reduces to (5.37) as follows.

In equation (5.4) write $n+1$ in place of n and subtract γ_p'' times the original equation. Add to the result $(\underline{I}_q \otimes \underline{\Gamma}_p'')$ times the original (5.4) and (5.37) results with $\underline{\Delta}_{n+1}''' = \underline{\Delta}_{n+2}'' - \gamma_p'' \underline{\Delta}_{n+1}'' + (\underline{I}_q \otimes \underline{\Gamma}_p'') \underline{\Delta}_{n+1}''$.

Remark

Observe the simplification when considering the classical methods.

The analysis which led to (5.11) is now valid if \underline{R} is replaced by \underline{R}^* and \underline{S} is replaced by \underline{S}^* . Thus, we have the following result.

Theorem 5.2

Under the assumptions of Lemma 5.4

$$[\underline{S}^*(E)\underline{P}(E) - \underline{Q}(E)\underline{R}^*(E)]\hat{\varphi}_{n-m} = \underline{S}^*(E)\underline{\Delta}_{n+1}' - \underline{Q}(E)\underline{\Delta}_{n+1}''' \quad (5.40)$$

and the associated stability polynomial is

$$\det[\underline{S}^*(\mu)\underline{P}(\mu) - \underline{Q}(\mu)\underline{R}^*(\mu)] \quad (5.41)$$

Some specific results appear below. Theorem 5.3 is of special interest since (in view of remarks of Hairer) we might choose \underline{A}'' with a sparse last row whilst \underline{A}' might be conventional.

Theorem 5.3

Consider the (extended) method $M(\Omega(\underline{A}'), \underline{e}; \Omega(\underline{A}''), \underline{e})$ applied to (5.2). The stability polynomial is

$$\det[\mu^2 \underline{Z} - \mu(\underline{Z} + \underline{E}_p + \eta h^2 \underline{A}' \underline{E}_p \underline{A}'') + \underline{E}_p] \quad (5.42)$$

$$\text{where } \underline{Z} \equiv \underline{Z}(\xi h, \eta h^2) = (\underline{I} - \xi h \underline{A} - \eta h^2 \underline{A}' \underline{A}''). \quad (5.43)$$

Proof

For the extended method $\underline{\gamma}' = \underline{\gamma}'' = \underline{e}$, $\underline{\Gamma}_p' = \underline{\Gamma}_p'' = \underline{E}_p$ and we set $\underline{A}_0' = \underline{A}_0'' = \underline{I}$, $\underline{A}_\ell' = \underline{A}_\ell'' = \underline{0}$ otherwise. From equation (5.32) - (5.35) and (5.38) with (5.39) we may write

$$\underline{P}(\mu) = [\underline{I} \otimes (\underline{I} - \xi h \underline{A}')] \mu^{m+1} - [\underline{I} \otimes \underline{E}_p] \mu^m$$

$$\underline{Q}(\mu) = -\eta h[\underline{I} \otimes \underline{A}'] \mu^{m+1}$$

$$\begin{aligned} \underline{R}^*(\mu) &= -h[\underline{I} \otimes ((\mu-1)\underline{I} + \underline{E}_p)][\underline{I} \otimes \underline{A}"] \mu^{m+1} \\ &= -h(\underline{I} \otimes \underline{A}'') \mu^{m+2} + h[\underline{I} \otimes (\underline{A}'' - \underline{E}_p \underline{A}'')] \mu^{m+1} \end{aligned}$$

$$\underline{S}^*(\mu) = (\mu-1) \mu^{m+1} (\underline{I} \otimes \underline{I})$$

Note that $\underline{S}^*(\mu)$ commutes with $\underline{Q}(\mu)$.

From (5.41) the stability polynomial is

$$\begin{aligned} &\det[\underline{S}^*(\mu) \underline{P}(\mu) - \underline{Q}(\mu) \underline{R}^*(\mu)] \\ &= [\underline{I} \otimes (\underline{I} - \xi h \underline{A}')] \mu^{2m+2} (\mu-1) - (\underline{I} \otimes \underline{E}_p) \mu^{2m+1} (\mu-1) - \\ &\quad - \eta h^2 (\underline{I} \otimes \underline{A}' \underline{A}'') \mu^{2m+3} + \eta h^2 [\underline{I} \otimes (\underline{A}' \underline{A}'' - \underline{A}' \underline{E}_p \underline{A}'')] \mu^{2m+2} \end{aligned}$$

Rearranging, we obtain

$$\begin{aligned} &[\underline{I} \otimes (\underline{I} - \xi h \underline{A}' - \eta h^2 \underline{A}' \underline{A}'')] \mu^{2m+3} \underline{I} \otimes [\underline{I} - \xi h \underline{A}' - \eta h^2 \underline{A}' \underline{A}'' + \underline{E}_p + \eta h^2 \underline{A}' \underline{E}_p \underline{A}''] \mu^{2m+2} + \\ &\quad (\underline{I} \otimes \underline{E}_p) \mu^{2m+1} \end{aligned}$$

from which (5.42) follows with \underline{Z} defined by (5.43).

Remark

Equation (5.42) may be expressed as

$$\begin{aligned} \det \frac{1}{\mu} [\lambda_0 \{ \mu(\underline{I} - \lambda_0 h \underline{A}') - \underline{E}_p \} \{ (\mu-1) \underline{I} + \underline{E}_p \} \{ \mu(\underline{I} - \lambda_1 h \underline{A}'') - \underline{E}_p \} \\ - \lambda_1 \{ \mu(\underline{I} - \lambda_1 h \underline{A}') - \underline{E}_p \} \{ (\mu-1) \underline{I} + \underline{E}_p \} \{ \mu(\underline{I} - \lambda_0 h \underline{A}'') - \underline{E}_p \}] \end{aligned} \quad (5.42')$$

If $\underline{A} = \underline{A}' = \underline{A}''$ then (5.42') reduces to a result of [3] which was also previously derived in Theorem 4.4.

The results of Theorems 5.4 and 5.5 which follow require the definitions of

$$\underline{\Psi}^*(\lambda h) = \underline{I} \otimes [\mu(\underline{I} - \lambda h \underline{A}') - \underline{E}_p] \mu^m \quad (5.44)$$

and of $\underline{\Psi}'(\lambda h)$ and $\underline{\Psi}''(\lambda h)$ obtained by inserting primes on \underline{A}_ℓ , \underline{B}_ℓ , \underline{A} and $\underline{\Gamma}_p$ in the definition

$$\underline{\Psi}(\lambda h) = \sum_{\ell=0}^{m+1} \{ \underline{A}_\ell \otimes [\mu(\underline{I} - \lambda h \underline{A}) - \underline{\Gamma}_p] - \lambda h \mu \underline{B}_\ell \otimes (\underline{E}_p - \underline{\Gamma}_p) \} \mu^{m-\ell} \quad (5.45)$$

Theorem 5.4

Consider the (extended/modified) method $M(\Omega(\underline{A}'), \underline{e}; \Omega[Q'', \underline{A}''], \underline{\gamma}'')$ applied to (5.2) and assume that the rules Q'' are simply block-reducible (or reducible). Then, if $\underline{\gamma}'' = \underline{e}$, the stability polynomial is (5.42).

For the general $\underline{\gamma}''$ the stability polynomial is

$$\det \frac{1}{\mu(\lambda_0 - \lambda_1)} [\lambda_0 \underline{\Psi}^*(\lambda_0 h) ((\mu - \gamma_p'')(\underline{I} \otimes \underline{I}) + (\underline{I} \otimes \underline{\Gamma}_p'')) \underline{\Psi}''(\lambda_1 h) - \lambda_1 \underline{\Psi}^*(\lambda_1 h) ((\mu - \gamma_p'')(\underline{I} \otimes \underline{I}) + (\underline{I} \otimes \underline{\Gamma}_p'')) \underline{\Psi}''(\lambda_0 h)] \quad (5.46)$$

Proof

For the general result, $\underline{\gamma}' = \underline{e}$, $\underline{\Gamma}_p' = \underline{E}_p$, \underline{V}_{nj}' are arbitrary and so we have $\underline{A}_0' = \underline{I}$ and $\underline{A}_\ell' = \underline{0}$ otherwise.

From equations (5.32) - (5.35) and (5.38) with (5.39) we may write

$$\begin{aligned} \underline{P}(\mu) &= [\underline{I} \otimes (\mu(\underline{I} - \xi h \underline{A}') - \underline{E}_p)] \mu^m \\ &= [\lambda_0 \underline{\Psi}^*(\lambda_0 h) - \lambda_1 \underline{\Psi}^*(\lambda_1 h)] / (\lambda_0 - \lambda_1) \\ \underline{Q}(\mu) &= -\eta h (\underline{I} \otimes \underline{A}') \mu^{m+1} \\ &= -\lambda_0 \lambda_1 [\underline{\Psi}^*(\lambda_0 h) - \underline{\Psi}^*(\lambda_1 h)] / (\lambda_0 - \lambda_1) \\ \underline{R}^*(\mu) &= -h [\underline{I} \otimes (\mu - \gamma_p'') \underline{I} + \underline{\Gamma}_p''] \sum_{\ell=0}^{m+1} [\underline{B}_\ell'' \otimes (\underline{E}_p - \underline{\Gamma}_p'') + \underline{A}_\ell'' \otimes \underline{A}''] \mu^{m+1-\ell} \\ &= \{ [\underline{I} \otimes (\mu - \gamma_p'') \underline{I} + \underline{\Gamma}_p''] (\underline{\Psi}''(\lambda_0 h) - \underline{\Psi}''(\lambda_1 h)) \} / (\lambda_0 - \lambda_1) \\ \underline{S}^*(\mu) &= [\underline{I} \otimes (\mu - \gamma_p'') \underline{I} + \underline{\Gamma}_p''] \sum_{\ell=0}^{m+1} [(\underline{A}_\ell'' \times \underline{I}) - (\underline{A}_{\ell-1}'' \otimes \underline{\Gamma}_p'')] \mu^{m+1-\ell} \\ &= [\underline{I} \otimes (\mu - \gamma_p'') \underline{I} + \underline{\Gamma}_p''] [\lambda_0 \underline{\Psi}''(\lambda_1 h) - \lambda_1 \underline{\Psi}''(\lambda_0 h)] / (\lambda_0 - \lambda_1) \end{aligned}$$

From (5.39) the alternative formulation for $\underline{S}^*(\mu)$ is

$$(\mu - \gamma_p'') \sum_{\ell=0}^{m+1} (\underline{A}_\ell'' \otimes \underline{I}) \mu^{m+1-\ell}. \text{ Thus, we observe that } \underline{S}^*(\mu) \text{ commutes}$$

with $\underline{Q}(\mu)$ and, in addition, $\underline{S}^*(\mu)$ commutes with $\underline{P}(\mu)$.

The stability polynomial is

$$\begin{aligned}
 & \det [\underline{S}^*(\mu) \underline{P}(\mu) - \underline{Q}(\mu) \underline{R}^*(\mu)] \\
 &= \det [\underline{P}(\mu) \underline{S}^*(\mu) - \underline{Q}(\mu) \underline{R}^*(\mu)] \\
 &= \det \left\{ [\lambda_0 \underline{\Psi}^*(\lambda_0 h) - \lambda_1 \underline{\Psi}^*(\lambda_1 h)] [\underline{I} \otimes (\mu - \gamma_p'') \underline{I} + \underline{\Gamma}_p^*] [\lambda_0 \underline{\Psi}''(\lambda_1 h) - \lambda_1 \underline{\Psi}''(\lambda_0 h)] \right. \\
 &+ \left. \lambda_1 \lambda_0 [\underline{\Psi}^*(\lambda_0 h) - \underline{\Psi}^*(\lambda_1 h)] [\underline{I} \otimes (\mu - \gamma_p'') \underline{I} + \underline{\Gamma}_p^*] [\underline{\Psi}''(\lambda_0 h) - \underline{\Psi}''(\lambda_1 h)] \right\} / (\lambda_0 - \lambda_1)^2
 \end{aligned}
 \tag{5.47}$$

On considering the expression

$$\begin{aligned}
 & \left\{ [\lambda_0 \underline{\Psi}^*(\lambda_0 h) - \lambda_1 \underline{\Psi}^*(\lambda_1 h)] [\underline{I} \otimes \underline{I}] [\lambda_0 \underline{\Psi}''(\lambda_1 h) - \lambda_1 \underline{\Psi}''(\lambda_0 h)] \right. \\
 &+ \left. \lambda_1 \lambda_0 [\underline{\Psi}^*(\lambda_0 h) - \underline{\Psi}^*(\lambda_1 h)] [\underline{I} \otimes \underline{I}] [\underline{\Psi}''(\lambda_0 h) - \underline{\Psi}''(\lambda_1 h)] \right\} (\mu - \gamma_p'')
 \end{aligned}$$

which is part of (5.47) and the similar expression with $\underline{I} \otimes \underline{\Gamma}_p''$ replacing $\underline{I} \otimes \underline{I}$ it is readily seen that (5.46) follows.

When $\underline{\gamma}'' = \underline{e}$, (5.46) becomes

$$\begin{aligned}
 & \lambda_0 \mu^{m+1} \sum_{\ell=0}^{m+1} \underline{A}_\ell'' \otimes (\mu^2 (\underline{I} - \lambda_0 h \underline{A}') (\underline{I} - \lambda_1 h \underline{A}'') - \mu [\underline{E}_p + (\underline{I} - \lambda_0 h \underline{A}') (\underline{I} - \lambda_1 h \underline{A}'') + \\
 & \quad \eta h^2 \underline{A}' \underline{E}_p \underline{A}''] + \underline{E}_p) \\
 & - \lambda_1 \mu^{m+1} \sum_{\ell=0}^{m+1} \underline{A}_\ell'' \otimes (\mu^2 (\underline{I} - \lambda_1 h \underline{A}') (\underline{I} - \lambda_0 h \underline{A}'') - \mu [\underline{E}_p + (\underline{I} - \lambda_1 h \underline{A}') (\underline{I} - \lambda_0 h \underline{A}'') + \\
 & \quad \eta h^2 \underline{A}' \underline{E}_p \underline{A}''] + \underline{E}_p)
 \end{aligned}$$

from which (5.42) follows.

The methods covered by the preceding results are of particular interest. The following is a general result.

Theorem 5.5

For the method $M(\Omega[Q', \underline{A}'], \underline{\gamma}'; \Omega[Q'', \underline{A}''], \underline{\gamma}'')$, where the rules Q'' are simply-block-reducible (or reducible), the determinant (5.41) reduces to (5.46) of Theorem 5.4 with $\underline{\Psi}^*(\lambda h)$ replaced by $\underline{\Psi}'(\lambda h)$.

5.5 Summary

In this chapter the intention was to investigate further the γ -modified Runge-Kutta methods. We have endeavoured to develop a means of constructing polynomials for the modified R-K methods when they are applied to the 'basic' test equation

$$f'(x) = \xi f(x) + \eta \int_0^x f(t) dt + d(x).$$

The analysis includes the stability polynomials for the classical R-K methods.

The general result (modified/modified method)

$M(\Omega[\underline{Q}', \underline{A}'], \underline{\gamma}'; \Omega[\underline{Q}'', \underline{A}''], \underline{\gamma}'')$ stated in Theorem 5.5 relies on the assumption that the rules \underline{Q}'' are simply-block-reducible (or reducible) as in general the presence of $\underline{\Gamma}_p''$ causes difficulties.

We demonstrate below the way in which earlier results in this chapter and those of chapter 4 may be derived from the general result.

1. The extended/modified method $M(\Omega(\underline{A}'), \underline{e}; \Omega(\underline{Q}'', \underline{A}''), \underline{\gamma}'')$ is derived by taking $\underline{\gamma}'' = \underline{e}$ in Theorem 5.5. This result is given in Theorem 5.4.
2. The extended/extended method $M(\Omega(\underline{A}'), \underline{e}; \Omega(\underline{A}''), \underline{e})$ is derived by taking $\underline{\gamma}' = \underline{\gamma}'' = \underline{e}$ in Theorem 5.5. This result is given in Theorem 5.3.
3. If, in addition to $\underline{\gamma}' = \underline{\gamma}'' = \underline{e}$ we take $\underline{A} = \underline{A}' = \underline{A}''$ we obtain $M(\Omega(\underline{A}), \underline{e}; \Omega(\underline{A}), \underline{e})$ which is the extended/extended result obtained in Theorem 4.4.
4. In Theorem 5.5 take $\underline{\gamma}' = \underline{e}$, $\underline{\gamma}'' = \underline{0}$ and $\underline{A} = \underline{A}' = \underline{A}''$ we obtain the associated stability polynomial

$$\det \frac{1}{\mu} \left\{ [\lambda_0 \underline{I} \otimes (\mu(\underline{I} - \lambda_0 h \underline{A}) - \underline{E}_p) \mu^{m+1}] \left[\sum_{\ell=0}^{m+1} (\underline{A}_{\ell}'' \otimes \mu(\underline{I} - \lambda_1 h \underline{A}) - \lambda_1 h \mu \hat{\underline{B}}_{\ell}'' \otimes \underline{E}_p) \mu^{m-\ell} \right] \right\}$$

$$\begin{aligned}
 & - [\lambda_1 \underline{I} \otimes \{\mu(\underline{I}-\lambda_1 h \underline{A}) - \underline{E}_p\} \mu^{m+1}] \left[\sum_{\ell=0}^{m+1} \{\underline{A}_\ell'' \otimes \mu(\underline{I}-\lambda_0 h \underline{A}) - \lambda_0 h \mu \hat{\underline{B}}_\ell'' \otimes \underline{E}_p\} \mu^{m-\ell} \right] \\
 & = \det (\lambda_0 - \lambda_1) \sum_{\ell=0}^{m+1} [\underline{A}_\ell'' \otimes \{(\underline{I} - \xi h \underline{A} - \eta h^2 \underline{A}^2) \mu - \underline{E}_p\} - \eta h^2 \mu \hat{\underline{B}}_\ell'' \otimes \underline{A} \underline{E}_p] \mu^{2m-\ell}.
 \end{aligned}$$

This corresponds to equation (4.72) in chapter 4 from which the result Theorem 4.6 is deduced for Runge Kutta/mixed methods using block reducible rules.

5. In Theorem 5.5 take $\underline{\gamma}' = \underline{\gamma}'' = \underline{0}$ and $\underline{A} = \underline{A}' = \underline{A}''$ and $\underline{Q}' = \underline{Q}'' = \underline{Q}$. We obtain the associated stability polynomial

$$\det \frac{1}{\mu} [\lambda_0 \underline{V}(\lambda_0 h) \underline{V}(\lambda_1 h) - \lambda_1 \underline{V}(\lambda_1 h) \underline{V}(\lambda_0 h)]$$

$$\text{where } \underline{V}(\lambda h) = \sum_{\ell=0}^{m+1} \{\underline{A}_\ell \otimes \mu (\underline{I} - \lambda h \underline{A}) - \lambda h (\underline{B}_\ell \otimes \underline{E}_p)\} \mu^{m-\ell}.$$

This is the result stated in (4.97) for the mixed/mixed method with block reducible rules.

REFERENCES

1. BAKER, C.T.H. Families of structured quadrature rules and some ramifications of reducibility. Int. Ser. Num. Math. 57 (ed. G. Hammerlin), (1982) pp.12-24.
2. BAKER, C.T.H., MAKROGLOU, A., & SHORT, E. Regions of stability in the numerical treatment of Volterra integro-differential equations. Siam J. Numer. Anal. 16 (1979) pp.809-910.
3. BAKER, C.T.H. & WILKINSON, J.C. Stability of Runge-Kutta methods applied to a basic integro-differential equation. Manuscript October 1981 (A revised version of Num.Anal.Tech.Rept. No. 50, University of Manchester, June 1980).
4. BAKER, C.T.H. & WILKINSON, J.C. On the construction of stability polynomials for modified R-K methods for Volterra integro-differential equations. Contribution to Treatment of Integral Equations by Numerical Methods (eds. BAKER, C.T.H. & MILLER, G.F.). Academic Press (1982). Also Num.Anal.Tech.Rept. No. 81, University of Manchester, September 1982.
5. van der HOUWEN, P.J. Convergence and stability results in R-K type methods for Volterra integral equations BIT 20 (1980) pp.375-377.
6. WOLKENFELT, P.H.M. The numerical analysis of reducible quadrature methods for Volterra integral and integro-differential equations. Acad. Proefschrift., Amsterdam (1981).